

# DOUBLE-BOSONISATION OF BRAIDED GROUPS AND THE CONSTRUCTION OF $U_q(g)$

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**Abstract** We introduce a quasitriangular Hopf algebra or ‘quantum group’  $U(B)$ , the *double-bosonisation*, associated to every braided group  $B$  in the category of  $H$ -modules over a quasitriangular Hopf algebra  $H$ , such that  $B$  appears as the ‘positive root space’,  $H$  as the ‘Cartan subalgebra’ and the dual braided group  $B^*$  as the ‘negative root space’ of  $U(B)$ . The choice  $B = f$  recovers Lusztig’s construction of  $U_q(g)$ , where  $f$  is Lusztig’s algebra associated to a Cartan datum; other choices give more novel quantum groups. As an application, our construction provides a canonical way of building up quantum groups from smaller ones by repeatedly extending their positive and negative root spaces by linear braided groups; we explicitly construct  $U_q(sl_3)$  from  $U_q(sl_2)$  by this method, extending it by the quantum-braided plane  $\mathbb{A}_q^2$ . We provide a fundamental representation of  $U(B)$  in  $B$ . A projection from the quantum double, a theory of double biproducts and a Tannaka-Krein reconstruction point of view are also provided.

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# 1 Introduction

The theory of braided groups or Hopf algebras in braided categories has been introduced by the author in 1989-1990 [1][2] as a more fundamental object underlying the theory of quantum groups. Braided planes, lines, matrices, Lie algebras, differentials and other constructions are now known in this braided-geometrical setting, developed in a series of papers by the author and collaborators; see [3][4] for reviews. The main idea of braided groups is that they are like Hopf algebras, with a diagonal or coproduct map  $B \rightarrow B \otimes B$ , but the tensor product here is not the usual commutative one; rather, it is a braided non-commutative tensor product. This ‘outer noncommutativity’ between two algebras is quite a different foundation for ‘braided geometry’ from the usual conception of non-commutative geometry based on the idea of a single ‘co-ordinate algebra’ becoming non-commutative. It is intended instead as a generalisation of supergeometry.

The reason that braided groups provide the foundation of a kind of  $q$ -deformed algebraic geometry is quite fundamental; It can be expected that the starting point of such a geometry should be the additive properties of  $\mathbb{R}^n$ , which means an additive or linear coproduct  $\Delta b = b \otimes 1 + 1 \otimes b$  on suitable generators. Such a coproduct is not interesting for an ordinary Hopf algebra since, being cocommutative, it belongs essentially to an enveloping algebra and not to a quantum group. However, for braided groups such a coproduct *is* compatible with  $q$ -deformation and, indeed, familiar examples such as the so-called quantum plane with relations  $yx = qxy$  have exactly such a linear coproduct or ‘coaddition’[5].

Apart from large classes of examples, there are also theorems (due to the author) which relate braided groups in certain braided categories to ordinary quantum groups[6][7]. This allows braided-group constructions to be used to obtain results about ordinary quantum groups. So far, an important application has been to the construction[5][8] of inhomogeneous quantum groups by ‘bosonisation’ of linear braided groups. The braided group appears as the ‘linear’ part of the inhomogeneous quantum group. The bosonisation construction[7][9] associates to every braided group  $B$  in the category of representations of a quasitriangular Hopf algebra  $H$  an ordinary Hopf algebra  $B \bowtie H$ . We recall the basic theory in the Preliminaries section 2.

The present paper extends this close connection between quantum groups and braided groups

with a new construction  $U(B)$  associated to the same data. This is a quasitriangular Hopf algebra in which our previous bosonisation  $H \bowtie B$  appears as ‘positive Borel subalgebra’ and the bosonisation  $B^* \bowtie H$  of the dual of  $B$  appears as ‘negative subalgebra’. The new part is the nontrivial cross relations between these two sub-Hopf algebras within  $U(B)$ . The background quantum group  $H$  (in the category of representations of which  $B, B^*$  live) plays the role of ‘Cartan subalgebra’. We introduce this construction in Section 3 and develop some of its basic properties. The triangular decomposition of  $U(B)$  into  $B^*, H, B$  is an intrinsic feature of our constructive definition. The most novel aspect is that apart from the general nature of our construction, the braided groups  $B^*$  and  $B$  need not be isomorphic; the positive and negative ‘roots’ are in general dual to each other rather than isomorphic.

Independently of the author’s development of braided groups and bosonisation, G. Lusztig in [10] introduced a novel construction of the quantum enveloping algebras  $U_q(g)$  of V.G. Drinfeld and M. Jimbo [11][12] associated to complex semisimple Lie algebras  $g$ . Although Lusztig does not use the formalism of braided groups, it is obvious that his algebra  $f$  with ‘coproduct’  $r : f \rightarrow f \otimes f$  could be viewed as an example of the  $q$ -braided type associated to a bilinear form, and that the resulting quantum Borel subalgebra  $U_q(b_+)$  could be viewed as its bosonisation. This is clear by comparison with the corresponding physics literature [13] where such  $q$ -braided groups and their bosonisation were studied in a quite different (physical) context. Such a view on Lusztig’s approach to  $U_q(b_+)$  has been pointed out most recently by M. Schauenberg at the Chicago AMS meeting in March 1995, and is one of the motivations for our new  $U(B)$  construction. In fact, we need something stronger, namely that  $f$  lives in the category generated by a certain *weakly quasitriangular* Hopf algebra, which we introduce. We are then able to cast Lusztig’s construction for all of  $U_q(g)$  into a braided setting.

In fact, we still require Lusztig’s elegant construction of  $f$  as the coradical of a bilinear pairing induced by the Cartan matrix datum, which provides the  $q$ -Serre relations of  $U_q(g)$  in his approach. But once we are given this as a (self-dual) braided group in a certain braided category, we can simply feed  $B = f = B^*$  into the abstract  $U(B)$  construction in the present paper and recover  $U_q(g)$  directly without the explicit proofs and calculations in [10]. This is demonstrated in detail in Section 4 and provides, we believe, a useful abstract setting for Lusztig’s approach. The fundamental Verma module representation in [10] is recovered now (in

an adjoint form) as a natural construction for an action of  $U(B)$  on  $B$  by ‘braided differentiation action’[14] of  $B^*$  and the ‘braided adjoint action’[15] of  $B$ .

The  $U(B)$  construction is also more general. We demonstrate an application of this in Section 5, where we begin with a central extension of the quantum group  $U_q(sl_2)$  in the role of ‘Cartan subalgebra’ and adjoin the so-called quantum  $B = \mathbb{A}_q^2$  to the positive root elements (and another copy to the negative roots) by means of our  $U(B)$  construction. The result is  $U_q(sl_3)$ , but constructed now in a novel way. We see that it is naturally represented on the quantum plane  $\mathbb{A}_q^2$  by braided differentiation, the braided adjoint action and  $U_q(su_2)$  rotations. The same construction works for  $U(V(R))$ , where  $V(R)$  is the linear braided group associated to suitable R-matrix data in  $M_n \otimes M_n$ . It adds one to the rank of the quantum group and  $n$  to the positive and negative roots. Some other, more physical, examples will be computed elsewhere as a construction of  $q$ -deformed conformal groups.

A second motivation for the  $U(B)$  construction is Drinfeld’s quantum double  $D(H)$  in [11]. Many constructions for Hopf algebras generalise easily (in a diagrammatic notation) to the setting of braided groups, so it is natural to ask for a braided-group version of Drinfeld’s double. While a braided group double cross product theory does exist, the example of the Drinfeld double (based on mutual coadjoint actions) appears to become ‘tangled up’; i.e., it works fine in a symmetric monoidal category but it encounters problems in a truly braided one. This is also true for even some of the simplest quantum group constructions, such as tensor products of braided groups. To overcome this problem we use the bosonisation procedure; since the bosonisation  $B \bowtie H$  is equivalent in a certain categorical sense to  $B$  (the ordinary modules of the former are in correspondence with the braided modules of the latter[7]), one can expect that the double of a braided group  $B$ , if it exists, should be closely related to the Drinfeld double of the bosonisation. We compute the latter in Appendix A and show that it projects onto  $U(B)$ . Hence  $U(B)$  could be regarded as some kind of bosonic (i.e. not braided) version of the ‘braided double’ of  $B$ ; it reduces to Drinfeld’s quantum double when  $H = k$ . The projection is also the means by which the quasitriangular structure of  $U(B)$  verified directly in Section 3, can be deduced, which is in analogy with the way that the quasitriangular structure of  $U_q(g)$  is obtained from the quantum double of the Borel subalgebra[11].

In Appendix B we show that  $U(B)$  can be viewed as special cases of a more general ‘double-

biproduct' construction, which we also introduce. Single bosonisations can be viewed as examples or single biproducts in the sense of [16], so this generalisation is a natural question. However, the double bosonisations remain the main examples of interest and their key properties do not come from this point of view. Finally, Appendix C provides a still different way of thinking about the complicated relations of the Hopf algebra  $U$ , namely as obtained by Tannaka-Krein reconstruction from a suitable category of braided crossed  $B - B^*$ -bimodules. One can also think of the latter as braided crossed  $B$ -modules in the sense of [17][18]. These are sufficiently complicated, however, that this is not a very convenient way to prove that  $U$  is a Hopf algebra, but provides an alternative viewpoint.

We work over a ground field  $k$ . With a little care one can work over a commutative ring just as well. We also note that our constructions will not be limited to finite-dimensional Hopf algebras.

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## 2 Preliminaries

Here we collect basic facts and notation from the theory of braided groups and their bosonisation, needed in Section 3 and the Appendix. For a more detailed review, see [3]. We also recall Lusztig's algebra  $f$  which is needed in Section 4. We begin with quantum groups in the sense of V.G. Drinfeld.

1. A *quasitriangular* Hopf algebra is  $(H, \Delta, \epsilon, S, \mathcal{R})$  where  $H$  is a unital algebra,  $\Delta : H \rightarrow H \otimes H$  and  $\epsilon : H \rightarrow k$  are algebra homomorphisms forming a coalgebra. This defines a bialgebra. In addition,  $S$  is the convolution-inverse of the identity  $H \rightarrow H$ , i.e. characterised by  $h_{(1)}Sh_{(2)} = \epsilon(h) = (Sh_{(1)})h_{(2)}$ , where we use the Sweedler notation[19]  $\Delta h = h_{(1)} \otimes h_{(2)}$  (summa-

tion understood). This defines a Hopf algebra. Finally,  $\mathcal{R} \in H \otimes H$  is invertible and obeys[11]

$$\begin{aligned} (\Delta \otimes \text{id})(\mathcal{R}) &= \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12} \\ \tau \circ \Delta &= \mathcal{R}(\Delta)\mathcal{R}^{-1}, \end{aligned} \tag{1}$$

where  $\mathcal{R}_{12} = \mathcal{R} \otimes 1 \in H^{\otimes 3}$  etc., and where  $\tau$  is the usual transposition map.

A *dual quasitriangular* bialgebra or Hopf algebra is  $(A, \mathcal{R})$  where  $A$  is a bialgebra or Hopf algebra and  $\mathcal{R} : A \otimes A \rightarrow k$  is a convolution-invertible linear map obeying the obvious dualisation of (1), namely

$$\begin{aligned} \mathcal{R} \circ (\cdot \otimes \text{id}) &= \mathcal{R}_{13} * \mathcal{R}_{23}, \quad \mathcal{R} \circ (\text{id} \otimes \cdot) = \mathcal{R}_{13} * \mathcal{R}_{12} \\ \cdot \circ \tau &= \mathcal{R} * \cdot * \mathcal{R}^{-1} \end{aligned} \tag{2}$$

in the convolution algebras  $\text{hom}(A \otimes A \otimes A, k)$  and  $\text{hom}(A \otimes A, A)$  respectively[20][21].

In between these two formulations of Drinfeld's ideas is an intermediate one[22] called a *weakly quasitriangular dual pair*. This is a pair  $(H, A)$  of Hopf algebras equipped with a duality pairing  $\langle \cdot, \cdot \rangle : H \otimes A \rightarrow k$  and convolution-invertible algebra/anti-coalgebra maps  $\mathcal{R}, \bar{\mathcal{R}} : A \rightarrow H$  obeying

$$\langle \bar{\mathcal{R}}(a), b \rangle = \langle \mathcal{R}^{-1}(b), a \rangle, \quad \forall a, b \in A, \quad \partial^R h = \mathcal{R} * (\partial^L h) * \mathcal{R}^{-1}, \quad \partial^R h = \bar{\mathcal{R}} * (\partial^L h) * \bar{\mathcal{R}}^{-1} \tag{3}$$

for all  $h \in H$ . Here  $*$  is the convolution product in  $\text{hom}(A, H)$  and  $(\partial^L h)(a) = \langle h_{(1)}, a \rangle h_{(2)}$ ,  $(\partial^R h)(a) = h_{(1)} \langle h_{(2)}, a \rangle$  are left and right 'differentiation operators' regarded as maps  $A \rightarrow H$  for fixed  $h$ . It is evident that given a dual pair of bialgebras or Hopf algebras,

$$\text{quasitriangularity} \Rightarrow \text{weak quasitriangularity} \Rightarrow \text{dual quasitriangularity}$$

by the appropriate evaluation using the duality pairing. It was shown by the author in 1989[23][22] that the dually paired bialgebras  $(A(R), \widetilde{U(R)})$  associated to a matrix solution  $R \in M_n \otimes M_n$  of the *Yang-Baxter equations*  $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$  are weakly quasitriangular, which means that  $A(R)$  is dual quasitriangular. Here  $M_n$  denotes  $n \times n$  matrices and  $R_{12} = R \otimes \text{id} \in M_n^{\otimes 3}$ , etc.

2. If  $H$  is a quasitriangular bialgebra or Hopf algebra then the categories  ${}_H\mathcal{M}$ ,  $\mathcal{M}_H$  of left modules and right modules are each braided. This means that for every two objects  $V, W$  there are functorial isomorphisms  $\Psi_{V,W} : V \otimes W \rightarrow W \otimes V$  which behave appropriately under  $\otimes$  of

modules. Explicitly, the braidings for left, right modules are

$$\Psi_{V,W}(v \otimes w) = \mathcal{R}^{(2)} \triangleright w \otimes \mathcal{R}^{(1)} \triangleright v, \quad \Psi_{V,W}(v \otimes w) = w \triangleleft \mathcal{R}^{(1)} \otimes v \triangleleft \mathcal{R}^{(2)} \quad (4)$$

for all  $v \in V, w \in W$ . Here  $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}$  is a notation for explicit components of  $\mathcal{R} \in H \otimes H$  (summation understood) and  $\triangleright, \triangleleft$  refer to left, right actions respectively.

It is a trivial matter to recast these formulae for the cases of weakly quasitriangular and dual quasitriangular bialgebras or Hopf algebras. Then the categories  ${}^A\mathcal{M}, \mathcal{M}^A$  of left, right comodules become braided[20][21].

3. An algebra in the category  $\mathcal{M}_H$  means a (right)  $H$ -module algebra, i.e. an algebra for which the structure maps intertwine (are covariant under) the action of  $H$ . A first result of the theory of braided groups is the observation (due to the author) that if  $B, C$  are such module algebras in a braided category (i.e. if  $H$  is quasitriangular) then there is an associative algebra  $B \underline{\otimes} C$ , the *braided tensor product algebra*[2] again in the category. The product rule is

$$(b \otimes c)(d \otimes e) = b \cdot \Psi_{C,B}(c \otimes d) \cdot e$$

where the output of  $\Psi$  is multiplied from the left by  $b$  and from the right by  $e$ . A bialgebra in the category means a unital algebra  $B$  equipped with algebra homomorphisms  $\underline{\Delta} : B \rightarrow B \underline{\otimes} B$ ,  $\underline{\epsilon} : B \rightarrow k$  forming a coalgebra and intertwining the action of  $H$ . A Hopf algebra means that in addition there is an intertwiner  $\underline{S} : B \rightarrow B$  which is the convolution inverse of the identity, i.e.  $(\underline{S}b_{(1)})b_{(2)} = \underline{\epsilon}(b) = b_{(1)}\underline{S}b_{(2)}$ , where  $\underline{\Delta} = b_{(1)} \otimes b_{(2)}$  denotes the braided coproduct. We use the term *braided group* to denote bialgebras or Hopf algebras in a braided category. They have been introduced and studied by the author [2][6][7], where basic properties such as

$$\underline{S}(bc) = \cdot \circ \Psi_{B,C}(\underline{S}b \otimes \underline{S}c) \quad (5)$$

are proven. There is also a further theory of quasitriangular braided groups (or braided quantum groups) which we do not need here; see [6][2]. The theory works in any braided category and we can easily read off the particular formulae for the other cases  ${}_H\mathcal{M}, {}^A\mathcal{M}$  and  $\mathcal{M}_A$  of interest.

Two braided groups  $C, B$  are said to be dually paired if there is an intertwiner  $\text{ev} : C \otimes B \rightarrow k$  such that  $\text{ev}(cd, b) = \text{ev}(d, b_{(1)})\text{ev}(c, b_{(2)})$  and  $\text{ev}(c, ab) = \text{ev}(c_{(2)}, a)\text{ev}(c_{(1)}, b)$  hold for all  $a, b \in B$

and  $c, d \in C$ . This is the natural *categorical duality* pairing[3]. In the finite-dimensional non-degenerate case we write  $C = B^*$ . In applications where we are finally interested in ordinary Hopf algebras, it is also useful to consider an *ordinary duality pairing*  $\langle \cdot, \cdot \rangle$  between braided groups  $C, B$  defined in the more usual way (without reversing the product or coproduct as for the categorical  $\text{ev}$ ). In this case, if  $B$  lives in  $\mathcal{M}_H$  then  $C$  lives naturally in  ${}_H\mathcal{M}$  and  $\langle h \triangleright c, b \rangle = \langle c, b \triangleleft h \rangle$  for all  $h \in H$  is the appropriate ‘covariance’ condition. In the finite-dimensional non-degenerate case we write  $C = B^*$ .

4. If  $B$  is a braided group in  ${}_H\mathcal{M}$  then its *bosonisation* is the Hopf algebra  $B \bowtie H$  defined as  $B \otimes H$  with product, coproduct and antipode[7]

$$\begin{aligned} (b \otimes h)(c \otimes g) &= bh_{(1)} \triangleright c \otimes h_{(2)}g, \quad \Delta(b \otimes h) = \underline{b}_{(1)} \otimes \mathcal{R}^{(2)}h_{(1)} \otimes \mathcal{R}^{(1)} \triangleright \underline{b}_{(2)} \otimes h_{(2)} \\ S(b \otimes h) &= (Sh_{(2)})u\mathcal{R}^{(1)} \triangleright \underline{S}b \otimes S(\mathcal{R}^{(2)}h_{(1)}); \quad u \equiv (S\mathcal{R}^{(2)})\mathcal{R}^{(1)} \end{aligned} \quad (6)$$

and tensor product unit and counit. The algebra structure is a smash product while the coalgebra is a smash coproduct by a particular coaction  $b \mapsto \mathcal{R}_{21} \triangleright b$  induced by the quasitriangular structure[24]. Bosonisations can be viewed[9] as a particular class of biproducts[16] (this class was not considered in any sense in [16], however). The right-handed version for  $B \in \mathcal{M}_H$  is  $H \bowtie B$  defined by

$$\begin{aligned} (h \otimes b)(g \otimes c) &= hg_{(1)} \otimes (b \triangleleft g_{(2)})c, \quad \Delta(h \otimes b) = h_{(1)} \otimes \underline{b}_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes h_{(2)}\mathcal{R}^{(2)} \otimes \underline{b}_{(2)} \\ S(h \otimes b) &= S(h_{(2)}\mathcal{R}^{(2)}) \otimes \underline{S}b \triangleleft \mathcal{R}^{(1)} v Sh_{(1)}, \quad v = \mathcal{R}^{(1)}S\mathcal{R}^{(2)}. \end{aligned} \quad (7)$$

The corresponding formulae for bosonisation in the comodule categories  ${}^A\mathcal{M}, \mathcal{M}^A$  are trivially obtained by the usual conversions of the module formulae;  $A \ltimes B$  and  $B \bowtie A$  have the smash coproduct coalgebra by the given coaction of  $A$  and smash product by the coaction induced by  $\mathcal{R}$ . For example, the first case (for  $B \in {}^A\mathcal{M}$ ) and its duality pairing with  $C \bowtie H$  is explicitly[25][8]

$$\begin{aligned} (a \otimes b)(d \otimes c) &= ad_{(1)} \otimes b^{(\bar{1})}c \langle \mathcal{R}(b^{(\bar{2})}), d_{(2)} \rangle, \quad \Delta(a \otimes b) = a_{(1)} \otimes \underline{b}_{(1)}^{(\bar{1})} \otimes a_{(2)}\underline{b}_{(1)}^{(\bar{2})} \otimes \underline{b}_{(2)} \\ \langle c \otimes h, a \otimes b \rangle &= \langle \mathcal{R}^{-(2)}h_{(1)}, a \rangle \text{ev}(\underline{S}^{-1}c, \mathcal{R}^{-(1)}h_{(2)} \triangleright b), \quad \langle S(c \otimes h), a \otimes b \rangle = \langle Sh, a \rangle \text{ev}(c, b). \end{aligned} \quad (8)$$

Moreover, if  $(H, A)$  is weakly quasitriangular then we can adapt the bosonisation formulae (6)–(7) to  $B \in \mathcal{M}^A, {}^A\mathcal{M}$  without dualisation; we define the action of  $H$  on  $B$  by evaluation against the given coaction of  $A$  and also replace  $\mathcal{R}_{21} \triangleright b$  by  $\mathcal{R}$  evaluated against the given coaction. These are all variants of the bosonisation construction in [7] in one form or another.



5. Let  $R \in M_n \otimes M_n$  obey the Yang-Baxter equations. We define[14] the *free braided vector algebra*  $V(R)$  to be the braided group with

$$\begin{aligned} B = k\langle e^i | i = 1, \dots, n \rangle, \quad \underline{\Delta}e^i &= e^i \otimes 1 + 1 \otimes e^i, \quad \epsilon e^i = 0 \\ \underline{S}e^i &= -e^i, \quad \Psi(e^i \otimes e^j) = \sum_{a,b} R^j_a{}^i_b e^a \otimes e^b. \end{aligned} \quad (9)$$

in the braided category of (left)  $A(R)$ -comodules. Here  $A(R)$  can be replaced by any dual-quasitriangular bialgebra or Hopf algebra, provided it induces the same braiding.

According to the braided-geometrical point of view (where  $e^i$  are like co-ordinates on a vector space), there is also a dually-paired *braided covector algebra*  $V^{\sim}(R)$ [14]. The version of it which has an ordinary duality pairing with  $V(R)$  is

$$\begin{aligned} D = k\langle f_i | i = 1, \dots, n \rangle, \quad \underline{\Delta}f_i &= f_i \otimes 1 + 1 \otimes f_i, \quad \underline{\epsilon}f_i = 0 \\ \underline{S}f_i &= -f_i, \quad \Psi(f_i \otimes f_j) = \sum_{a,b} f_a \otimes f_b R^a{}_i{}^b{}_j \end{aligned} \quad (10)$$

in the braided category of right  $A(R)$ -comodules. The pairing is defined by  $\langle f_j, e^i \rangle = \delta_j^i$ , where  $\delta$  is the Kronecker delta function. The categorical dual  $C$  has the opposite product and coproduct; it looks the same on generators but has  $\Psi(f_i \otimes f_j) = \sum_{a,b} f_a \otimes f_b R^a{}_j{}^b{}_i$ . According to (a version of) the theory in [14] there is a left action of  $C$  on  $V(R)$  by *braided differentiation*

$$\partial_i : V(R) \rightarrow V(R), \quad \partial_i(e^{i_1} \dots e^{i_m}) = [m; R]_{i_1 j_2 \dots j_m}^{i_1 \dots i_m} e^{j_2} \dots e^{j_m}, \quad (11)$$

where

$$[m; R] = \text{id} + (PR)_{12} + (PR)_{23}(PR)_{12} + \dots (PR)_{m-1 \ m} \dots (PR)_{12} \in M_n^{\otimes m} \quad (12)$$

is the *braided integer matrix*[14]. Here  $P$  denotes the permutation matrix in  $M_n \otimes M_n$  and  $(PR)_{12} = PR \otimes \text{id}$ , etc. Then the categorical pairing between  $C$  and  $B$  is  $\text{ev}(c(f), b) = \epsilon(c(\partial)b)$ . Equivalently, the ordinary duality pairing between  $D$  and  $B$  is

$$\langle f_{j_1} \dots f_{j_m}, e^{i_1} \dots e^{i_r} \rangle = \delta_m^r [m; R]_{j_1 \dots j_m}^{i_1 \dots i_m}, \quad [m; R]! = [m; R]_{1 \dots m} [m-1; R]_{2 \dots m} \dots [2; R]_{m-1 \ m} \quad (13)$$

where the numerical suffices denote as usual the embedding of our matrices in the corresponding positions in  $M_n^{\otimes m}$ . Equivalently,  $\partial_i b$  is characterised by  $\underline{\Delta}b = e^i \otimes \partial_i b +$  terms which do not consist of  $e^i$  in the first tensor factor. There are also right braided differentials  $\overleftarrow{\partial}_i$ .

Various properties are proven about such braided differential operators  $V(R) \rightarrow V(R)$  in [14]. In particular, we introduce a corresponding exponential

$$\exp_R = \sum_m e^{i_1} \cdots e^{i_m} \otimes f_{j_1} \cdots f_{j_m} ([m; R]!)_{i_1 \cdots i_m}^{j_1 \cdots j_m} \quad (14)$$

as a formal power-series in the matrix entries of  $R$  with coefficients in  $B \otimes D$ . This assumes that the pairing is non-degenerate, i.e. that the  $[m, R]!$  matrices are all invertible. In many applications, such as the braided Taylor's theorem[14], only a finite number of terms from  $\exp_R$  contribute.

When the pairing  $\langle , \rangle$  is degenerate, one can add to further relations to both sides until it becomes non-degenerate. In particular[14] if there is a second matrix  $R' \in M_n \otimes M_n$  obeying certain relations with  $R$ , one can add to  $B, D$  the quadratic relations

$$e^i e^j = \sum_{a,b} R'^j{}_a{}^i{}_b e^a e^b, \quad f_i f_j = \sum_{a,b} f_b f_a R'^a{}_i{}^b{}_j \quad (15)$$

to  $B$  and  $D$ . If both  $R, R'$  are  $q$ -deformations of the identity matrix then by genericity arguments we will know that the resulting quotients  $V(R', R)$  and  $V^\sim(R', R)$  are non-degenerately paired. This is the case for the  $sl_n$  quantum planes  $\mathbb{A}_q^n$  in Section 5, where  $R' = q^{-2}R$ . The  $\partial_i$  descend to these quotients[14].

Lusztig's algebra  $f$  in [10] can be viewed as a braided group quotient of the free braided plane  $V(R)$  with  $R^i{}_j{}^k{}_l = q^{i \cdot k} \delta^i{}_j \delta^k{}_l$ , where  $0 \neq q \in k$  and  $i \cdot j$  are the components of a bilinear form over  $\mathbb{Z}$ . In Lusztig's example the braided differentiation operators are denoted  ${}_i r$  and  $r_i$ , and the radical of the pairing is far from quadratic. The braided coproduct  $\underline{\Delta}$  is denoted  $r$  in [10]. The braided antipode  $\underline{S}$  does not appear explicitly. See Section 4 for further details on how to view Lusztig's algebra  $f$  as a braided group.

### 3 Quasitriangular Hopf algebra $U(B)$ associated to a braided group

Let  $H$  be a quasitriangular Hopf algebra and  $B$  a braided group in the category  $\mathcal{M}_H$ . Let  $C$  be a braided group in  $\mathcal{M}_H$  which is dual to  $B$  in the sense of an categorical duality pairing  $\text{ev}$ . Equivalently,  $D = C^{\text{op/op}}$  is a Hopf algebra in  ${}_H \mathcal{M}$  which is dual to  $B$  in the sense of an ordinary duality pairing  $\langle , \rangle$  which is  $H$ -bicovariant as explained above. We construct in this

section an associated quasitriangular Hopf algebra  $U(B)$ . We suppose that  $C$  (or, equivalently,  $D$ ) has invertible braided-antipode. We denote by  $\bar{H}$  the same Hopf algebra as  $H$  but equipped with the quasitriangular structure  $\bar{\mathcal{R}} = \mathcal{R}_{21}^{-1}$ .

**Lemma 3.1** [3] *If  $D$  is a braided group in  ${}_H\mathcal{M}$  then  $D^{\text{cop}}$  defined by the same algebra, unit and counit as  $D$  and the opposite coproduct  $\Psi_{D,D}^{-1} \circ \underline{\Delta}$  and antipode  $\underline{S}^{-1}$ , is a braided group in  $_{\bar{H}}\mathcal{M}$ .*

**Proof** A diagrammatic proof is given in [3]. The lemma is also easily checked directly from the axioms (and is in fact the reason that the naive concept of braided-cocommutativity for braided groups  $\underline{\Delta} = \Psi^{-1} \circ \underline{\Delta}$  does not make sense; see [1]).  $\square$

We see that  $\bar{C} \equiv D^{\text{cop}} = (C^{\text{op/op}})^{\text{cop}}$  is a Hopf algebra in  $_{\bar{H}}\mathcal{M}$ . We denote its coproduct explicitly by  $\bar{\Delta} = c_{(\bar{1})} \otimes c_{(\bar{2})}$  and its antipode by  $\bar{S}$ . According to the bosonisation theory recalled in the preliminaries, we immediately have two Hopf algebras  $H \bowtie B$  and  $\bar{C} \bowtie \bar{H}$ .  $H$  itself is a sub-Hopf algebra of each.

**Theorem 3.2** *There is a unique Hopf algebra structure  $U = U(\bar{C}, H, B)$  on  $\bar{C} \otimes H \otimes B$  such that  $H \bowtie B$  and  $\bar{C} \bowtie \bar{H}$  are sub-Hopf algebras by the canonical inclusions and*

$$bc = (\mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})}) \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} (b_{(\bar{2})} \triangleleft \mathcal{R}_2^{-(1)}) \langle \mathcal{R}_1^{(1)} \triangleright c_{(\bar{1})}, b_{(\bar{1})} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} c_{(\bar{3})}, b_{(\bar{3})} \triangleleft \mathcal{R}_2^{-(2)} \rangle$$

for all  $b \in B, c \in \bar{C}$  viewed in  $U$ . Here  $\mathcal{R}_1, \mathcal{R}_2$  etc. are distinct copies of the quasitriangular structure  $\mathcal{R}$  of  $H$ .

We will prove this via a series of lemmas. We begin by proving associativity. Note first that if there is an associative product as stated, then it is given uniquely by

$$(c \otimes h \otimes b) \cdot (d \otimes g \otimes a) = c(h_{(1)} \mathcal{R}_1^{(2)} \triangleright d_{(\bar{2})}) \otimes h_{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} g_{(1)} \otimes (b_{(\bar{2})} \triangleleft \mathcal{R}_2^{-(1)} g_{(2)}) a \\ \langle \mathcal{R}_1^{(1)} \triangleright d_{(\bar{1})}, b_{(\bar{1})} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} d_{(\bar{3})}, b_{(\bar{3})} \triangleleft \mathcal{R}_2^{-(2)} \rangle \quad (16)$$

for all  $c, d \in \bar{C}$ ,  $a, b \in B$  and  $h, g \in H$ : Because  $\bar{C} \bowtie \bar{H}$  and  $H \bowtie B$  are subalgebras, we know that a general product has the form

$$(chb) \cdot (dga) = \sum c(hd_i) R_i(b_i g) a = \sum c(h_{(1)} \triangleright d_i) h_{(2)} R_i g_{(1)} (b_i \triangleleft g_{(2)}) a \quad (17)$$

if  $bd = \sum d_i R_i b_i$  say, where  $d_i \in \bar{C}$ ,  $R_i \in H$  and  $b_i \in B$  are all viewed in  $U$  in the canonical way.

We take the right hand side as the definition of the product of general elements.

**Lemma 3.3** *The map (16) is an associative product on  $U = \bar{C} \otimes H \otimes B$ .*

**Proof** It is enough to prove associativity in the special case  $(a \cdot (chb)) \cdot d = a \cdot ((chb) \cdot d)$  for all  $a, b \in B$ ,  $c, d \in \bar{C}$  and  $h \in H$ , viewed in  $\bar{C} \otimes H \otimes B$  in the canonical way. One can then deduce the general case by breaking down products in the form (17) and using that  $C$  is a left  $H$ -module algebra and  $B$  a right  $H$ -module algebra.

To prove the special case, we compute:

$$\begin{aligned}
(a \cdot (chb)) \cdot d &= \left( \mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})} \otimes \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} h_{(1)} \otimes (a_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(2)} h_{(2)}) b \right) \cdot (d \otimes 1 \otimes 1) \\
&= \langle \mathcal{R}_1^{(1)} \triangleright c_{(\bar{1})}, a_{(\underline{1})} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} c_{(\bar{3})}, a_{(\underline{3})} \triangleleft \mathcal{R}_2^{-(2)} \rangle \\
&= (\mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})}) ((\mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} h_{(1)})_{(1)} \mathcal{R}_3^{(2)} \triangleright d_{(\bar{2})}) \otimes (\mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} h_{(1)})_{(2)} \mathcal{R}_4^{(2)} \mathcal{R}_3^{-(1)} \\
&\otimes \left( (a_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)} h_{(2)}) b \right)_{(\underline{2})} \triangleleft \mathcal{R}_4^{-(1)} \langle \mathcal{R}_1^{(1)} \triangleright c_{(\bar{1})}, a_{(\underline{1})} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} c_{(\bar{3})}, a_{(\underline{3})} \triangleleft \mathcal{R}_2^{-(2)} \rangle \\
&\langle \mathcal{R}_3^{(1)} \triangleright d_{(\bar{1})}, \left( (a_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)} h_{(2)}) b \right)_{(\underline{1})} \triangleleft \mathcal{R}_4^{(1)} \rangle \langle \mathcal{R}_3^{-(2)} \triangleright \bar{S} d_{(\bar{3})}, \left( (a_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)} h_{(2)}) b \right)_{(\underline{3})} \triangleleft \mathcal{R}_4^{-(2)} \rangle
\end{aligned}$$

from the definition (16). Next, we use that  $\underline{\Delta}$  and hence  $(\text{id} \otimes \underline{\Delta}) : B \rightarrow B \otimes B \otimes B$  is an algebra homomorphism to the braided tensor product algebra, i.e.

$$(\text{id} \otimes \underline{\Delta}) \circ \underline{\Delta}(ab) = a_{(\underline{1})} (b_{(\underline{1})} \triangleleft \mathcal{R}_8^{(1)} \mathcal{R}_9^{(1)}) \otimes (a_{(\underline{2})} \triangleleft \mathcal{R}_9^{(2)}) (b_{(\underline{2})} \triangleleft \mathcal{R}_{10}^{(1)}) \otimes (a_{(\underline{3})} \triangleleft \mathcal{R}_8^{(2)} \mathcal{R}_{10}^{(2)}) b_{(\underline{3})}$$

where  $\mathcal{R}_8, \dots, \mathcal{R}_{10}$  are fresh copies of  $\mathcal{R}$  distinct from others to be used below. We also use that the product and coproduct of  $B$  are covariant under  $H$ . The first gives us the action  $\triangleleft \mathcal{R}_4^{-(2)}$  etc on products of elements of  $B$ , and the second gives us the coproduct  $(\text{id} \otimes \underline{\Delta}) \circ \underline{\Delta}(a \triangleleft h)$  etc. We then use the axioms (1) to convert all coproducts of  $\mathcal{R}$  to products of  $\mathcal{R}$ , suitably numbered. We use covariance of  $\langle \cdot, \cdot \rangle$  to move  $\mathcal{R}_1^{(1)} \triangleright$  etc. in its first input to  $\triangleleft \mathcal{R}_1^{(1)}$  etc. in its second input. We arrive by these steps at the expression:

$$\begin{aligned}
(a \cdot (chb)) \cdot d &= (\mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})}) (\mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} h_{(1)} \mathcal{R}_3^{(2)} \mathcal{R}_{11}^{(2)} \triangleright d_{(\bar{2})}) \\
&\otimes \mathcal{R}_5^{(2)} \mathcal{R}_5^{-(1)} h_{(2)} \mathcal{R}_4^{(2)} \mathcal{R}_{10}^{(2)} \mathcal{R}_3^{-(1)} \mathcal{R}_8^{-(1)} \otimes (a_{(\underline{3})} \triangleleft \mathcal{R}_6^{-(1)} h_{(4)} \mathcal{R}_9^{(2)} \mathcal{R}_4^{-(1)} \mathcal{R}_9^{-(1)}) (b_{(\underline{2})} \triangleleft \mathcal{R}_{10}^{-(1)}) \\
&\langle c_{(\bar{1})}, a_{(\underline{1})} \triangleleft \mathcal{R}_5^{(1)} \mathcal{R}_2^{(1)} \mathcal{R}_1^{(1)} \rangle \langle \bar{S} c_{(\bar{3})}, a_{(\underline{5})} \triangleleft \mathcal{R}_7^{-(2)} \mathcal{R}_6^{-(2)} \mathcal{R}_2^{-(2)} \mathcal{R}_5^{-(2)} \mathcal{R}_1^{-(2)} \rangle \\
&\langle d_{(\bar{1})}, (a_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)} h_{(3)} \mathcal{R}_4^{(1)} \mathcal{R}_3^{(1)}) (b_{(\underline{1})} \triangleleft \mathcal{R}_8^{(1)} \mathcal{R}_9^{(1)} \mathcal{R}_{10}^{(1)} \mathcal{R}_{11}^{(1)}) \rangle \\
&\langle \bar{S} d_{(\bar{3})}, (a_{(\underline{4})} \triangleleft \mathcal{R}_7^{-(1)} h_{(5)} \mathcal{R}_8^{(2)} \mathcal{R}_4^{-(2)} \mathcal{R}_3^{-(2)}) (b_{(\underline{3})} \triangleleft \mathcal{R}_{10}^{-(2)} \mathcal{R}_9^{-(2)} \mathcal{R}_8^{-(2)}) \rangle.
\end{aligned}$$

We next use the braided duality pairing between  $B, C$ , which between  $B, \bar{C}$  takes the form

$$\langle d, ab \rangle = \langle d_{(\bar{2})}, a \triangleleft \mathcal{R}^{(2)} \rangle \langle d_{(\bar{1})}, b \triangleleft \mathcal{R}^{(1)} \rangle, \quad \langle \bar{S} d, ab \rangle = \langle \bar{S} d_{(\bar{1})}, a \rangle \langle \bar{S} d_{(\bar{2})}, b \rangle, \quad \forall d \in \bar{C}, \quad a, b \in B. \quad (18)$$

Hence,

$$\begin{aligned}
(a.(chb)).d &= (\mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})})(\mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} h_{(1)} \mathcal{R}_3^{(2)} \mathcal{R}_{11}^{(2)} \triangleright d_{(\bar{3})}) \\
&\otimes \mathcal{R}_5^{(2)} \mathcal{R}_5^{-(1)} h_{(2)} \mathcal{R}_4^{(2)} \mathcal{R}_{10}^{(2)} \mathcal{R}_3^{-(1)} \mathcal{R}_8^{-(1)} \otimes (a_{(\underline{3})} \triangleleft \mathcal{R}_6^{-(1)} h_{(4)} \mathcal{R}_9^{(2)} \mathcal{R}_4^{-(1)} \mathcal{R}_9^{-(1)}) (b_{(\underline{2})} \triangleleft \mathcal{R}_{10}^{-(1)}) \\
&\langle c_{(\bar{1})}, a_{(\underline{1})} \triangleleft \mathcal{R}_5^{(1)} \mathcal{R}_2^{(1)} \mathcal{R}_1^{(1)} \rangle \langle \bar{S} c_{(\bar{3})}, a_{(\underline{5})} \triangleleft \mathcal{R}_7^{-(2)} \mathcal{R}_6^{-(2)} \mathcal{R}_2^{-(2)} \mathcal{R}_5^{-(2)} \mathcal{R}_1^{-(2)} \rangle \\
&\langle d_{(\bar{2})}, a_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)} h_{(3)} \mathcal{R}_4^{(1)} \mathcal{R}_3^{(1)} \mathcal{R}_{12}^{(2)} \rangle \langle d_{(\bar{1})}, b_{(\underline{1})} \triangleleft \mathcal{R}_8^{(1)} \mathcal{R}_9^{(1)} \mathcal{R}_{10}^{(1)} \mathcal{R}_{11}^{(1)} \mathcal{R}_{12}^{(1)} \rangle \\
&\langle \bar{S} d_{(\bar{4})}, a_{(\underline{4})} \triangleleft \mathcal{R}_7^{-(1)} h_{(5)} \mathcal{R}_8^{(2)} \mathcal{R}_4^{-(2)} \mathcal{R}_3^{-(2)} \rangle \langle \bar{S} d_{(\bar{5})}, b_{(\underline{3})} \triangleleft \mathcal{R}_{10}^{-(2)} \mathcal{R}_9^{-(2)} \mathcal{R}_8^{-(2)} \rangle.
\end{aligned}$$

On the other side, we similarly compute

$$\begin{aligned}
a \cdot ((chb) \cdot d) &= (1 \otimes 1 \otimes a) \cdot \left( c(h_{(1)} \mathcal{R}_1^{(2)} \triangleright d_{(\bar{2})}) \otimes h_{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} \otimes b_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)} \right) \\
&\langle \mathcal{R}_1^{(1)} \triangleright d_{(\bar{1})}, b_{(\underline{1})} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} d_{(\bar{3})}, b_{(\underline{3})} \triangleleft \mathcal{R}_2^{-(2)} \rangle \\
&= \mathcal{R}_3^{(2)} \triangleright (c(h_{(1)} \mathcal{R}_1^{(2)} \triangleright d_{(\bar{2})}))_{(\bar{2})} \otimes \mathcal{R}_4^{(2)} \mathcal{R}_3^{-(1)} (h_{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)})_{(1)} \\
&\otimes \left( a_{(\underline{2})} \triangleleft \mathcal{R}_4^{-(1)} (h_{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)})_{(2)} \right) (b_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)}) \langle \mathcal{R}_1^{(1)} \triangleright d_{(\bar{1})}, b_{(\underline{1})} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{-(2)} \triangleright \bar{S} d_{(\bar{3})}, b_{(\underline{3})} \triangleleft \mathcal{R}_2^{-(2)} \rangle \\
&\langle \mathcal{R}_3^{(1)} \triangleright (c(h_{(1)} \mathcal{R}_1^{(2)} \triangleright d_{(\bar{2})}))_{(\bar{1})}, a_{(\underline{1})} \triangleleft \mathcal{R}_4^{(1)} \rangle \langle \mathcal{R}_3^{-(2)} \triangleright \bar{S} (c(h_{(1)} \mathcal{R}_1^{(2)} \triangleright d_{(\bar{2})}))_{(\bar{3})}, a_{(\underline{3})} \triangleleft \mathcal{R}_4^{-(2)} \rangle \\
&= (\mathcal{R}_3^{(2)} \triangleright c_{(\bar{2})})(\mathcal{R}_5^{(2)} \mathcal{R}_{10}^{(2)} \mathcal{R}_7^{-(1)} h_{(2)} \mathcal{R}_6^{(2)} \triangleright d_{(\bar{3})}) \otimes \mathcal{R}_4^{(2)} \mathcal{R}_9^{(2)} \mathcal{R}_3^{-(1)} \mathcal{R}_9^{-(1)} h_{(4)} \mathcal{R}_2^{(2)} \mathcal{R}_1^{-(1)} \\
&\otimes (a_{(\underline{3})} \triangleleft \mathcal{R}_4^{-(1)} \mathcal{R}_8^{-(1)} h_{(5)} \mathcal{R}_8^{(2)} \mathcal{R}_5^{-(1)}) (b_{(\underline{2})} \triangleleft \mathcal{R}_2^{-(1)}) \\
&\langle d_{(\bar{1})}, b_{(\underline{1})} \triangleleft \mathcal{R}_8^{(1)} \mathcal{R}_2^{(1)} \mathcal{R}_7^{(1)} \mathcal{R}_6^{(1)} \mathcal{R}_1^{(1)} \rangle \langle \bar{S} d_{(\bar{5})}, b_{(\underline{3})} \triangleleft \mathcal{R}_2^{-(2)} \mathcal{R}_5^{-(2)} \mathcal{R}_1^{-(2)} \rangle \\
&\langle c_{(\bar{1})}, a_{(\underline{1})} \triangleleft \mathcal{R}_4^{(1)} \mathcal{R}_5^{(1)} \mathcal{R}_3^{(1)} \rangle \langle d_{(\bar{2})}, a_{(\underline{2})} \triangleleft \mathcal{R}_9^{(1)} \mathcal{R}_{10}^{(1)} \mathcal{R}_6^{-(1)} h_{(1)} \mathcal{R}_1^{(2)} \rangle \\
&\langle \bar{S} c_{(\bar{3})}, a_{(\underline{5})} \triangleleft \mathcal{R}_8^{-(2)} \mathcal{R}_9^{-(2)} \mathcal{R}_{10}^{-(2)} \mathcal{R}_7^{-(2)} \mathcal{R}_6^{-(2)} \rangle \langle \bar{S} d_{(\bar{4})}, a_{(\underline{4})} \triangleleft \mathcal{R}_4^{-(2)} \mathcal{R}_3^{-(2)} \mathcal{R}_{10}^{-(1)} h_{(3)} \mathcal{R}_7^{(2)} \rangle,
\end{aligned}$$

using in order: the definition (16), the homomorphism property of the iterated braided-coproduct of  $\bar{C}$  in the form

$$(\text{id} \otimes \bar{\Delta}) \circ \bar{\Delta}(cd) = c_{(\bar{1})}(\mathcal{R}_{11}^{-(1)} \mathcal{R}_6^{-(1)} \triangleright d_{(\bar{1})}) \otimes (\mathcal{R}_{11}^{-(2)} \triangleright c_{(\bar{2})})(\mathcal{R}_7^{-(1)} \triangleright d_{(\bar{2})}) \otimes (\mathcal{R}_7^{-(2)} \mathcal{R}_6^{-(2)} \triangleright c_{(\bar{3})}) d_{(\bar{3})},$$

the covariance of the product and coproduct of  $\bar{C}$  and of  $\langle , \rangle$ , the quasitriangularity axiom (1) and the evaluation pairing between  $B, \bar{C}$  in the form

$$\langle cd, b \rangle = \langle c, b_{(\underline{1})} \rangle \langle d, b_{(\underline{2})} \rangle, \quad \langle \bar{S}(cd), b \rangle = \langle \bar{S}d, b_{(\underline{1})} \triangleleft \mathcal{R}^{-(1)} \rangle \langle \bar{S}c, b_{(\underline{2})} \triangleleft \mathcal{R}^{-(2)} \rangle, \quad \forall c, d \in \bar{C}, b \in B. \quad (19)$$

These steps are similar to the proof of associativity of the usual Drinfeld quantum double, except that now there are several copies of  $\mathcal{R}$  inserted at various points (arising from

the braiding in the categories in which  $B, \bar{C}$  live). It remains to show that these are correctly placed. First, we use the quantum Yang-Baxter equations for  $\mathcal{R}$  in  $H^{\otimes 3}$ , applied to  $\mathcal{R}_{10}, \mathcal{R}_6^{-1}, \mathcal{R}_7^{-1}$ . Then we are able to use the quasicocommutativity axiom (1) in the form  $\mathcal{R}_{10}(h_{(1)} \otimes h_{(2)}) = (h_{(2)} \otimes h_{(1)})\mathcal{R}_{10}$ . We make this kind of rearrangement three more times: we use the QYBE applied to  $\mathcal{R}_3^{-1}, \mathcal{R}_9^{-1}, \mathcal{R}_{10}^{-1}$  and reverse the order of  $\mathcal{R}_3^{-1}(h_{(4)} \otimes h_{(3)})$ ; we then use the QYBE applied to  $\mathcal{R}_9, \mathcal{R}_9^{-1}, \mathcal{R}_6^{-1}$  and reverse the order of  $\mathcal{R}_9(h_{(2)} \otimes h_{(3)})$ ; we use the QYBE applied to  $\mathcal{R}_4^{-1}, \mathcal{R}_8^{-1}, \mathcal{R}_{10}^{-1}$  and reverse the order of  $\mathcal{R}_4^{-1}(h_{(5)} \otimes h_{(4)})$ . The result is

$$\begin{aligned}
a \cdot ((chb) \cdot d) &= (\mathcal{R}_3^{(2)} \triangleright c_{(\bar{2})}) (\mathcal{R}_5^{(2)} \mathcal{R}_7^{-1} h_{(1)} \mathcal{R}_{10}^{(2)} \mathcal{R}_6^{(2)} \triangleright d_{(\bar{3})}) \\
&\otimes \mathcal{R}_4^{(2)} \mathcal{R}_9^{-1} h_{(3)} \mathcal{R}_9^{(2)} \mathcal{R}_3^{-1} \mathcal{R}_2^{(2)} \mathcal{R}_1^{-1} \otimes (a_{(\underline{3})} \triangleleft \mathcal{R}_8^{-1} h_{(4)} \mathcal{R}_4^{-1} \mathcal{R}_8^{(2)} \mathcal{R}_5^{-1}) (b_{(\underline{2})} \triangleleft \mathcal{R}_2^{-1}) \\
&\langle d_{(\bar{1})}, b_{(\underline{1})} \triangleleft \mathcal{R}_8^{(1)} \mathcal{R}_2^{(1)} \mathcal{R}_7^{(1)} \mathcal{R}_6^{(1)} \mathcal{R}_1^{(1)} \rangle \langle \bar{S} d_{(\bar{5})}, b_{(\underline{3})} \triangleleft \mathcal{R}_2^{-2} \mathcal{R}_5^{-2} \mathcal{R}_1^{-2} \rangle \\
&\langle c_{(\bar{1})}, a_{(\underline{1})} \triangleleft \mathcal{R}_4^{(1)} \mathcal{R}_5^{(1)} \mathcal{R}_3^{(1)} \rangle \langle d_{(\bar{2})}, a_{(\underline{2})} \triangleleft \mathcal{R}_6^{-1} h_{(3)} \mathcal{R}_9^{(1)} \mathcal{R}_{10}^{(1)} \mathcal{R}_1^{(2)} \rangle \\
&\langle \bar{S} c_{(\bar{3})}, a_{(\underline{5})} \triangleleft \mathcal{R}_{10}^{-2} \mathcal{R}_8^{-2} \mathcal{R}_6^{-2} \mathcal{R}_9^{-2} \mathcal{R}_7^{-2} \rangle \langle \bar{S} d_{(\bar{4})}, a_{(\underline{4})} \triangleleft \mathcal{R}_{10}^{-1} h_{(5)} \mathcal{R}_4^{-2} \mathcal{R}_3^{-2} \mathcal{R}_7^{(2)} \rangle.
\end{aligned}$$

This coincides with our previous expression for  $(a \cdot (chb)) \cdot d$  in all respects except the order of some copies of  $\mathcal{R}$ . We use the QYBE several times more (i.e. we identify the corresponding braids as generated by the appropriate braidings  $\Psi$ ) to see that the two expressions are in fact equal. Hence we have an associative product on  $U = \bar{C} \otimes H \otimes B$ . It is clear that  $(1 \otimes 1 \otimes 1)$  a unit element for it.  $\square$

Next we note that if the coproducts of  $\bar{C} \rtimes \bar{H}$  and  $H \ltimes B$  extend to a coproduct on  $U$ , the latter must be given by

$$\Delta(c \otimes h \otimes b) = c_{(\bar{1})} \otimes \mathcal{R}^{-1} h_{(1)} \otimes b_{(\underline{1})} \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{-2} \triangleright c_{(\bar{2})} \otimes h_{(2)} \mathcal{R}^{(2)} \otimes b_{(\underline{2})} \quad (20)$$

for all  $c \in \bar{C}$ ,  $b \in B$ ,  $h \in H$ . This is  $(\Delta(ch))\Delta b$  or  $(\Delta c)\Delta(hb)$  in  $U \otimes U$  as computed from (16).

**Lemma 3.4** *The map (20) makes the algebra  $U$  into a bialgebra.*

**Proof** It is enough to prove that  $\Delta(bc) = (\Delta b) \cdot (\Delta c)$  for all  $b \in B$  and  $c \in \bar{C}$ , where the product in  $U \otimes U$  is the usual tensor product one. After proving this, we compute  $\Delta((chb) \cdot (dga))$  using the definition (17) of a general product, and then use that  $\bar{C}$  is a braided group in the category of left  $\bar{H}$ -modules,  $B$  a braided group in the category of right  $H$ -modules and the quasicocommutativity axiom (1) to obtain  $(\Delta c)(\Delta h)(\Delta(bd))(\Delta g)\Delta(a)$ . Since  $\bar{C} \rtimes \bar{H}$  and  $H \ltimes B$  are

subalgebras and the product restricted to them is already known (by the bosonisation theorem) to form a bialgebra, we obtain  $(\Delta(ch))(\Delta b)(\Delta d)(\Delta(ga)) = (\Delta(chb))(\Delta(dga))$ .

It remains to prove the special case. We compute

$$\begin{aligned}
\Delta(b \cdot c) &= \Delta \left( \mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})} \otimes \mathcal{R}_2^{(2)} \mathcal{R}_1^{- (1)} \otimes \underline{b}_{(2)} \triangleleft \mathcal{R}_2^{- (1)} \right) \\
&\quad \langle \mathcal{R}_1^{(1)} \triangleright c_{(\bar{1})}, \underline{b}_{(1)} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{- (2)} \triangleright \bar{S}c_{(\bar{3})}, \underline{b}_{(3)} \triangleleft \mathcal{R}_2^{- (2)} \rangle \\
&= (\mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})})_{(\bar{1})} \otimes \mathcal{R}_3^{- (1)} (\mathcal{R}_2^{(2)} \mathcal{R}_1^{- (1)})_{(1)} \otimes (\underline{b}_{(2)} \triangleleft \mathcal{R}_2^{- (1)})_{(1)} \triangleleft \mathcal{R}_3^{(1)} \\
&\quad \otimes \mathcal{R}_3^{- (2)} \triangleright (\mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})})_{(\bar{2})} \otimes (\mathcal{R}_2^{(2)} \mathcal{R}_1^{- (1)})_{(2)} \mathcal{R}_3^{(2)} \otimes (\underline{b}_{(2)} \triangleleft \mathcal{R}_2^{- (1)})_{(2)} \\
&\quad \langle \mathcal{R}_1^{(1)} \triangleright c_{(\bar{1})}, \underline{b}_{(1)} \triangleleft \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{- (2)} \triangleright \bar{S}c_{(\bar{3})}, \underline{b}_{(3)} \triangleleft \mathcal{R}_2^{- (2)} \rangle \\
&= \mathcal{R}_1^{(2)} \triangleright c_{(\bar{2})} \otimes \mathcal{R}_3^{- (1)} \mathcal{R}_2^{(2)} \mathcal{R}_1^{- (1)} \otimes \underline{b}_{(2)} \triangleleft \mathcal{R}_2^{- (1)} \mathcal{R}_3^{(1)} \\
&\quad \otimes \mathcal{R}_3^{- (2)} \mathcal{R}_4^{(2)} \triangleright c_{(\bar{3})} \otimes \mathcal{R}_5^{(2)} \mathcal{R}_4^{- (1)} \mathcal{R}_3^{(2)} \otimes \underline{b}_{(3)} \triangleleft \mathcal{R}_5^{- (1)} \\
&\quad \langle c_{(\bar{1})}, \underline{b}_{(1)} \triangleleft \mathcal{R}_5^{(1)} \mathcal{R}_2^{(1)} \mathcal{R}_4^{(1)} \mathcal{R}_1^{(1)} \rangle \langle \bar{S}c_{(\bar{4})}, \underline{b}_{(4)} \triangleleft \mathcal{R}_5^{- (2)} \mathcal{R}_2^{- (2)} \mathcal{R}_4^{- (2)} \mathcal{R}_1^{- (2)} \rangle
\end{aligned}$$

using the product (16) and the definition (20) of the map  $\Delta$ . We used the covariance of the coproducts of  $\bar{C}$  and  $B$  and their pairing  $\langle \cdot, \cdot \rangle$ , and then wrote all coproducts of  $\mathcal{R}$  as products using (1).

On the other side, we compute

$$\begin{aligned}
(\Delta b) \cdot (\Delta c) &= (1 \otimes 1 \otimes \underline{b}_{(1)} \triangleleft \mathcal{R}_1^{(1)}) \cdot (c_{(\bar{1})} \otimes \mathcal{R}_1^{- (1)} \otimes 1) \otimes (1 \otimes \mathcal{R}_1^{(2)} \otimes \underline{b}_{(2)}) \cdot (\mathcal{R}_1^{- (2)} \triangleright c_{(\bar{2})} \otimes 1 \otimes 1) \\
&= \mathcal{R}_2^{(2)} \triangleright c_{(\bar{1})}(\bar{2}) \otimes \mathcal{R}_3^{(2)} \mathcal{R}_2^{- (1)} \mathcal{R}_1^{- (1)}_{(1)} \otimes (\underline{b}_{(1)} \triangleleft \mathcal{R}_1^{(1)})_{(2)} \triangleleft \mathcal{R}_3^{- (1)} \mathcal{R}_1^{- (1)}_{(2)} \\
&\quad \langle \mathcal{R}_2^{(1)} \triangleright c_{(\bar{1})}(\bar{1}), (\underline{b}_{(1)} \triangleleft \mathcal{R}_1^{(1)})_{(1)} \triangleleft \mathcal{R}_3^{(1)} \rangle \langle \mathcal{R}_2^{- (2)} \triangleright \bar{S}c_{(\bar{1})}(\bar{3}), (\underline{b}_{(1)} \triangleleft \mathcal{R}_1^{(1)})_{(3)} \triangleleft \mathcal{R}_3^{- (2)} \rangle \\
&\quad \otimes \mathcal{R}_1^{(2)}_{(1)} \mathcal{R}_4^{(2)} \triangleright (\mathcal{R}_1^{- (2)} \triangleright c_{(\bar{2})})_{(\bar{2})} \otimes \mathcal{R}_1^{(2)}_{(2)} \mathcal{R}_5^{(2)} \mathcal{R}_4^{- (1)} \otimes \underline{b}_{(2)(2)} \triangleleft \mathcal{R}_5^{- (1)} \\
&\quad \langle \mathcal{R}_4^{(1)} \triangleright (\mathcal{R}_1^{- (2)} \triangleright c_{(\bar{2})})_{(\bar{1})}, \underline{b}_{(2)(1)} \triangleleft \mathcal{R}_5^{(1)} \rangle \langle \mathcal{R}_4^{- (2)} \triangleright \bar{S}(\mathcal{R}_1^{- (2)} \triangleright c_{(\bar{2})})_{(\bar{3})}, \underline{b}_{(2)(3)} \triangleleft \mathcal{R}_5^{- (2)} \rangle \\
&= \mathcal{R}_2^{(2)} \triangleright c_{(\bar{2})} \otimes \mathcal{R}_3^{(2)} \mathcal{R}_2^{- (2)} \mathcal{R}_1^{- (1)} \mathcal{R}_7^{- (1)} \mathcal{R}_8^{- (1)} \otimes \underline{b}_{(2)} \triangleleft \mathcal{R}_9^{(1)} \mathcal{R}_7^{(1)} \mathcal{R}_3^{- (1)} \mathcal{R}_6^{- (1)} \mathcal{R}_9^{- (1)} \mathcal{R}_{10}^{- (1)} \\
&\quad \otimes \mathcal{R}_1^{(2)} \mathcal{R}_7^{(2)} \mathcal{R}_8^{(2)} \mathcal{R}_4^{(2)} \mathcal{R}_9^{- (2)} \mathcal{R}_7^{- (2)} \triangleright c_{(\bar{5})} \otimes \mathcal{R}_6^{(2)} \mathcal{R}_9^{(2)} \mathcal{R}_{10}^{(2)} \mathcal{R}_5^{(2)} \mathcal{R}_4^{- (1)} \otimes \underline{b}_{(5)} \triangleleft \mathcal{R}_5^{- (1)} \\
&\quad \langle c_{(\bar{1})}, \underline{b}_{(1)} \triangleleft \mathcal{R}_6^{(1)} \mathcal{R}_1^{(1)} \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \rangle \langle \bar{S}c_{(\bar{6})}, \underline{b}_{(6)} \triangleleft \mathcal{R}_5^{- (2)} \mathcal{R}_4^{- (2)} \mathcal{R}_{10}^{- (2)} \mathcal{R}_8^{- (2)} \rangle \\
&\quad \langle \bar{S}c_{(\bar{3})}, \underline{b}_{(3)} \triangleleft \mathcal{R}_{10}^{(1)} \mathcal{R}_8^{(1)} \mathcal{R}_3^{- (2)} \mathcal{R}_2^{- (2)} \rangle \langle c_{(\bar{4})}, \underline{b}_{(4)} \triangleleft \mathcal{R}_5^{(1)} \mathcal{R}_4^{(1)} \mathcal{R}_6^{- (2)} \mathcal{R}_1^{- (2)} \rangle
\end{aligned}$$

using the usual induced product in  $U \otimes U$ , the definition (20) of the map  $\Delta$  and the covariance of the coproducts of  $\bar{C}$ ,  $B$  and their pairing  $\langle \cdot, \cdot \rangle$ . As usual, we write coproducts of  $\mathcal{R}$  as products

via (1). We next use (1) in reverse to recognise the element of  $H \otimes H$  acting on  $\underline{b}_{(3)} \otimes \underline{b}_{(4)}$  as in the image of the coproduct of  $H$ . Then (19) tells us that

$$\begin{aligned}
(\Delta b) \cdot (\Delta c) &= \mathcal{R}_2^{(2)} \triangleright_{c_{(2)}} \otimes \mathcal{R}_3^{(2)} \mathcal{R}_1^{-(1)} \mathcal{R}_7^{-(1)} \mathcal{R}_8^{-(1)} \otimes \underline{b}_{(2)} \triangleleft \mathcal{R}_9^{(1)} \mathcal{R}_7^{(1)} \mathcal{R}_6^{-(1)} \mathcal{R}_9^{-(1)} \mathcal{R}_{10}^{-(1)} \\
&\quad \otimes \mathcal{R}_1^{(2)} \mathcal{R}_7^{(2)} \mathcal{R}_4^{(2)} \mathcal{R}_9^{-(2)} \mathcal{R}_7^{-(2)} \triangleright_{c_{(4)}} \otimes \mathcal{R}_6^{(2)} \mathcal{R}_9^{(2)} \mathcal{R}_5^{(2)} \mathcal{R}_4^{-(1)} \otimes \underline{b}_{(4)} \triangleleft \mathcal{R}_5^{-(1)} \\
&\quad \langle c_{(1)}, \underline{b}_{(1)} \triangleleft \mathcal{R}_6^{(1)} \mathcal{R}_1^{(1)} \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \rangle \langle \bar{S} c_{(6)}, \underline{b}_{(5)} \triangleleft \mathcal{R}_5^{-(2)} \mathcal{R}_4^{-(2)} \mathcal{R}_{10}^{-(2)} \mathcal{R}_8^{-(2)} \rangle \\
&\quad \langle (\bar{S} c_{(3)}) c_{(4)}, \underline{b}_{(3)} \triangleleft \mathcal{R}_5^{(1)} \mathcal{R}_4^{(1)} \mathcal{R}_6^{-(2)} \mathcal{R}_1^{-(2)} \rangle \\
&= \mathcal{R}_2^{(2)} \triangleright_{c_{(2)}} \otimes \mathcal{R}_3^{(2)} \mathcal{R}_7^{-(1)} \mathcal{R}_8^{-(1)} \otimes \underline{b}_{(2)} \triangleleft \mathcal{R}_9^{(1)} \mathcal{R}_{10}^{-(1)} \otimes \mathcal{R}_1^{(2)} \mathcal{R}_7^{-(2)} \triangleright_{c_{(3)}} \otimes \mathcal{R}_6^{(2)} \mathcal{R}_9^{(2)} \mathcal{R}_4^{-(2)} \\
&\quad \otimes \underline{b}_{(3)} \triangleleft \mathcal{R}_5^{-(1)} \langle c_{(1)}, \underline{b}_{(1)} \triangleleft \mathcal{R}_6^{(1)} \mathcal{R}_1^{(1)} \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \rangle \langle \bar{S} c_{(4)}, \underline{b}_{(4)} \triangleleft \mathcal{R}_5^{-(2)} \mathcal{R}_4^{-(2)} \mathcal{R}_{10}^{-(2)} \mathcal{R}_8^{-(2)} \rangle,
\end{aligned}$$

where we use the axiom for the braided-antipode  $\bar{S}$  of  $\bar{C}$ . We also cancelled  $\mathcal{R}_7 \mathcal{R}_9^{-1}$  from the final expression.

The resulting two expressions differ only by the order of the  $\mathcal{R}$  factors. They are in fact equal on using the QYBE several times, i.e. equating the action of two braids generated by the corresponding braidings  $\Psi$ .

This completes the proof that the map  $\Delta : U \rightarrow U \otimes U$  is an algebra homomorphism. It is coassociative since this is known for  $\bar{C} \rtimes \bar{H}$  and  $H \ltimes B$  separately. The tensor product of the counits of  $\bar{C}$ ,  $H$  and  $B$  clearly provides the counit for  $\Delta$ ; hence we have a bialgebra  $U$ .  $\square$

We define the antipode of  $U$  by  $S(chb) = (Sb) \cdot (S(ch))$  or  $(S(hb)) \cdot Sc$  in terms of the known antipodes  $S$  of  $H \ltimes B$  or  $\bar{C} \rtimes \bar{H}$ . We do not need its exact form; it can be computed from the formulae in the Preliminaries and the product (16).

**Lemma 3.5** *The antipodes of  $\bar{C} \rtimes \bar{H}$  and  $H \ltimes B$  extend to an antipode  $S : U \rightarrow U$ .*

**Proof** The two extensions  $S(chb) = (Sb) \cdot (S(ch))$  and  $S(chb) = ((Shb)) \cdot Sc$  are equal because both are equal (using associativity in  $U$ , as proven above) to  $(Sb)(Sh)(Sc)$ . We use that the restriction  $S(ch)$  is the antipode of  $\bar{C} \rtimes \bar{H}$ , which is known to be a Hopf algebra[7] so that  $S$  is antimultiplicative on  $ch$ . Likewise, the restriction  $S(hb)$  is the antipode of  $H \ltimes B$  and is therefore antimultiplicative as well. Working in  $U$ , we now write  $\Delta c = c_{(1)} \otimes c_{(2)}$  (the bosonised coproduct) where  $c_{(1)} \in \bar{C} \rtimes \bar{H}$  and  $c_{(2)} \in \bar{C}$ . Likewise,  $\Delta b = b_{(1)} \otimes b_{(2)}$  where  $b_{(1)} \in B$  and  $b_{(2)} \in H \ltimes B$ . Since  $U$  is a bialgebra, as proven above, we have  $(S(chb)_{(1)}) \cdot (chb)_{(2)} = (S((c_{(1)} h_{(1)}) \cdot b_{(1)})) \cdot (c_{(2)} h_{(2)}) \cdot b_{(2)} =$



$(Sb_{(1)}) \cdot ((S(c_{(1)}h_{(1)}))c_{(2)}h_{(2)}) \cdot b_{(2)} = (\epsilon c)(Sb_{(1)})b_{(2)} = (\epsilon c)(\epsilon b)$ . We used that the restriction of  $S : U \rightarrow U$  to  $\bar{C} \bowtie \bar{H}$  is its antipode, and then that the restriction of  $S$  to  $H \bowtie B$  is the antipode for that. Similarly for the proof of antipode axiom on the other side.  $\square$

This completes the proof of Theorem 3.2. When  $B$  is finite-dimensional we take  $C = B^*$  the categorical dual (i.e.  $D = B^*$  the ordinary dual) and write  $U = U(B)$ , the *double-bosonisation* of  $B$ . In this case we have a canonical element or coevaluation for the duality pairing. As we have seen in (14), when the braided coproducts are linear ones this coevaluation plays the role of exponential. See [26] where this point of view is developed further in general (diagrammatic) terms as part of a braided Fourier theory. With this in mind, we write

$$\exp_B = \sum e_a \otimes f^a, \quad \text{e}\bar{\text{x}}\text{p}_B = \sum f^a \otimes \underline{S}e_a \quad (21)$$

where  $\{e_a\}$  is a basis of  $B$  with dual basis  $\{f^a\}$ , and  $\underline{S}$  is the braided antipode of  $B$ . Here  $\text{e}\bar{\text{x}}\text{p}_B = \exp_{B21}^{-1}$  is the transposed inverse in the usual (unbraided) tensor product algebra  $B^* \otimes B$ . A specific example of the same formalism is the braided Fourier transform[27] which plays a role in conformal field theory.

**Proposition 3.6** *If  $B$  is finite-dimensional then its double-bosonisation  $U(B)$  is quasitriangular, with quasitriangular structure*

$$\mathcal{R}_U = \text{e}\bar{\text{x}}\text{p}_B \cdot \mathcal{R} = \sum (f^a \otimes \mathcal{R}_2^{(1)} \mathcal{R}_1^{(1)} \otimes 1) \otimes (\mathcal{R}_1^{(2)} \otimes \underline{S}e_a \triangleleft \mathcal{R}_2^{(2)})$$

where the product  $\cdot$  is in  $U \otimes U$ .

**Proof** We verify the quasitriangular structure directly. In the appendix we introduce a (non-trivial) projection from a suitable Drinfeld quantum double, which can also be used to obtain  $\mathcal{R}_U$ .

From a categorical point of view, the exponential is the coevaluation for the pairing between  $B^*$  and  $B$ . In terms of the structure of  $\bar{C} = (B^*)^{\text{cop}}$  and  $B$ , the pairing (18) corresponds to the coevaluation property

$$(\bar{\Delta} \otimes \text{id})\text{e}\bar{\text{x}}\text{p}_B = f^a \otimes f^b \otimes (\underline{S}e_a)(\underline{S}e_b) = \text{e}\bar{\text{x}}\text{p}_{B13}\text{e}\bar{\text{x}}\text{p}_{B23}. \quad (22)$$

Here the numerical suffices denote positions in the tensor product  $B^* \otimes B \otimes B$ . Likewise, (19) corresponds to the coevaluation property

$$(\text{id} \otimes \underline{\Delta})\text{exp}_B = (\mathcal{R}^{(2)} \triangleright f^b)(\mathcal{R}^{(1)} \triangleright f^a) \otimes \underline{S}e_a \otimes \underline{S}e_b. \quad (23)$$

This is equivalent to  $(\text{id} \otimes \underline{\Delta})\text{exp}_B = \text{exp}_{B12}\text{exp}_{B13}$  given the braided-antimultiplicativity of the braided antipode  $\underline{S}$ . The covariance of the pairing corresponds in terms of the coevaluation to  $h \triangleright f^a \otimes e_a = f^a \otimes e_a \triangleleft h$  for all  $h \in H$ .

Using (22), and since the coproduct of  $U$  restricts on  $H$  to its coproduct and  $\mathcal{R} \in H \otimes H$  already obeys (1), have

$$\begin{aligned} (\Delta \otimes \text{id})\mathcal{R}_U &= (\Delta f^a \otimes \underline{S}e_a)\mathcal{R}_{13}\mathcal{R}_{23} \\ &= (f^a_{(1)}\mathcal{R}^{-(1)} \otimes \mathcal{R}^{-(2)} \triangleright f^a_{(2)} \otimes \underline{S}e_a)\mathcal{R}_{13}\mathcal{R}_{23} \\ &= \text{exp}_{B13}(\mathcal{R}^{-(1)} \otimes f^b \otimes \underline{S}e_b \triangleleft \mathcal{R}^{-(2)})\mathcal{R}_{13}\mathcal{R}_{23} \\ &= \text{exp}_{B13}\mathcal{R}_{13}(\mathcal{R}_1^{-(1)}\mathcal{R}_2^{-(1)} \otimes f^b \otimes \mathcal{R}_1^{-(2)}(\underline{S}e_b \triangleleft \mathcal{R}_2^{-(2)}))\mathcal{R}_{13}\mathcal{R}_{23} \\ &= \text{exp}_{B13}\mathcal{R}_{13}\text{exp}_{B23}\mathcal{R}_{23} = \mathcal{R}_{U13}\mathcal{R}_{U23}. \end{aligned}$$

Products here are in  $U$  or its tensor powers. The second equality applies the coproduct of  $U$  from (20) or  $\bar{C} \rtimes \bar{H}$ . The third equality is (22) and covariance of the coevaluation. The fourth inserts  $\mathcal{R}_{13}\mathcal{R}_{13}^{-1}$  and allows us (via (1)) to recognise the product in the middle section as  $\text{exp}_{B23}\mathcal{R}_{13}^{-1}$  according to the relations of  $H \bowtie B \subseteq U$ .

Similarly, using (23), we have

$$\begin{aligned} (\text{id} \otimes \Delta)\mathcal{R}_U &= (f^a \otimes \Delta \underline{S}e_a)\mathcal{R}_{13}\mathcal{R}_{12} \\ &= (f^a \otimes (\underline{S}e_a)_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}(\underline{S}e_a)_{(2)})\mathcal{R}_{13}\mathcal{R}_{12} \\ &= (f^b(\mathcal{R}_2^{(1)}\mathcal{R}_1^{(1)} \triangleright f^a) \otimes \underline{S}e_a \otimes \mathcal{R}_1^{(2)}(\underline{S}e_b \triangleleft \mathcal{R}_2^{(2)}))\mathcal{R}_{13}\mathcal{R}_{12} \\ &= \text{exp}_{B13}((\mathcal{R}_1^{(1)} \triangleright f^a)\mathcal{R}_2^{(1)} \otimes \underline{S}e_a \otimes \mathcal{R}_1^{(2)}\mathcal{R}_2^{(2)})\mathcal{R}_{12} \\ &= \text{exp}_{B13}\mathcal{R}_{13}\text{exp}_{B12}\mathcal{R}_{12} = \mathcal{R}_{U13}\mathcal{R}_{U12}. \end{aligned}$$

The second equality is the coproduct of  $U$  or  $H \bowtie B$ , the third is (23) and covariance of the coevaluation. The fourth recognises the relations of  $H \bowtie B \subseteq U$  and the fifth recognises the relations of  $\bar{C} \rtimes \bar{H} \subseteq U$ .

This proves the first two parts of (1) for  $\mathcal{R}_U$ . Next, we compute for all  $b \in B \subseteq U$ ,

$$\begin{aligned}
(\Delta^{\text{op}}b)\mathcal{R}_U &= \mathcal{R}_2^{(2)}b_{(2)}f^a\mathcal{R}_1^{(1)} \otimes (b_{(1)}\triangleleft\mathcal{R}_2^{(1)})(\underline{S}e_a)\mathcal{R}_1^{(2)} \\
&= \mathcal{R}_2^{(2)}(\mathcal{R}_3^{(2)}\triangleright f^a_{(\bar{2})})\mathcal{R}_4^{(2)}\mathcal{R}_1^{-(1)}(b_{(2)(2)}\triangleleft\mathcal{R}_2^{-(1)})\mathcal{R}_1^{(1)} \otimes (b_{(1)}\triangleleft\mathcal{R}_2^{(1)})(\underline{S}e_a)\mathcal{R}_1^{(2)} \\
&\quad \langle \mathcal{R}_3^{(1)}\triangleright f^a_{(\bar{1})}, b_{(2)(1)}\triangleleft\mathcal{R}_4^{(1)} \rangle \langle \mathcal{R}_1^{-(2)}\triangleright \bar{S}f^a_{(\bar{3})}, b_{(2)(3)}\triangleleft\mathcal{R}_2^{-(2)} \rangle \\
&= \mathcal{R}_2^{(2)}(\mathcal{R}_3^{(2)}\triangleright f^b)\mathcal{R}_4^{(2)}\mathcal{R}_1^{-(1)}(b_{(3)}\triangleleft\mathcal{R}_2^{-(1)})\mathcal{R}_1^{(1)} \otimes (b_{(1)}\triangleleft\mathcal{R}_2^{(1)})(\underline{S}e_a)(\underline{S}e_b)(\underline{S}e_c)\mathcal{R}_1^{(2)} \\
&\quad \langle f^a, b_{(2)}\triangleleft\mathcal{R}_4^{(1)}\mathcal{R}_3^{(1)} \rangle \langle \bar{S}f^c, b_{(4)}\triangleleft\mathcal{R}_2^{-(2)}\mathcal{R}_1^{-(2)} \rangle \\
&= (\mathcal{R}_2^{(2)}\mathcal{R}_3^{(2)}\triangleright f^b)\mathcal{R}_5^{(2)}\mathcal{R}_4^{(2)}\mathcal{R}_1^{-(1)}(b_{(3)}\triangleleft\mathcal{R}_2^{-(1)})\mathcal{R}_1^{(1)} \\
&\quad \otimes (b_{(1)}\triangleleft\mathcal{R}_5^{(1)}\mathcal{R}_2^{(1)})(\underline{S}b_{(2)}\triangleleft\mathcal{R}_4^{(1)}\mathcal{R}_3^{(1)})(\underline{S}e_b)(b_{(4)}\triangleleft\mathcal{R}_2^{-(2)}\mathcal{R}_1^{-(2)})\mathcal{R}_1^{(2)} \\
&= (\mathcal{R}_3^{(2)}\triangleright f^b)\mathcal{R}_4^{(2)}\mathcal{R}_1^{-(1)}(b_{(3)}\triangleleft\mathcal{R}_2^{-(1)})\mathcal{R}_1^{(1)} \otimes ((b_{(1)}(\underline{S}b_{(2)}\triangleleft\mathcal{R}_4^{(1)}\mathcal{R}_3^{(1)})(\underline{S}e_b)(b_{(4)}\triangleleft\mathcal{R}_2^{-(2)}\mathcal{R}_1^{-(2)})\mathcal{R}_1^{(2)}) \\
&= f^b\mathcal{R}_1^{-(1)}(b_{(1)}\triangleleft\mathcal{R}_2^{-(1)})\mathcal{R}_1^{(1)} \otimes (\underline{S}e_b)(b_{(2)}\triangleleft\mathcal{R}_2^{-(2)}\mathcal{R}_1^{-(2)})\mathcal{R}_1^{(2)} \\
&= f^bb_{(1)}\mathcal{R}_1^{-(1)}\mathcal{R}_2^{(1)}\mathcal{R}_1^{(1)} \otimes (\underline{S}e_b)\mathcal{R}_1^{(2)}(b_{(2)}\triangleleft\mathcal{R}_1^{-(2)}\mathcal{R}_2^{(2)}) = f^bb_{(1)}\mathcal{R}^{(1)} \otimes (\underline{S}e_b)\mathcal{R}^{(2)}b_{(2)} \\
&= f^b\mathcal{R}_1^{(1)}(b_{(1)}\triangleleft\mathcal{R}_2^{(1)}) \otimes (\underline{S}e_b)\mathcal{R}_1^{(2)}\mathcal{R}_2^{(2)}b_{(2)} = \mathcal{R}_U\Delta b
\end{aligned}$$

Here the first equality is the definition of the coproduct of  $U$  or  $H\bowtie B$  (transposed) and  $\mathcal{R}_U$ . Products are in  $U \otimes U$ . The second equality uses the relations in Theorem 3.2 to reorder  $b_{(2)}f^a$ . The third equality uses (22) in iterated form. The fourth equality evaluates the canonical element as an identity mapping, and also uses the relations of  $\bar{C}\bowtie\bar{H} \subseteq U$  to move  $\mathcal{R}_2^{(2)}$  to the right. The fifth equality uses (1) in reverse and covariance of the product of  $B$ . The sixth equality is the axioms for the braided-antipode  $\underline{S}$  of  $B$ . The seventh equality recognises the relations of  $H\bowtie B$  as  $b_{(3)}\mathcal{R}_2^{-(1)}$ , and also uses these relations to move  $\mathcal{R}_1^{(2)}$  to the left. The result is  $\mathcal{R}_U\Delta b$  by a further application of these relations of  $H\bowtie H$ .

The proof of  $(\Delta^{\text{op}}c)\mathcal{R}_U = \mathcal{R}_U\Delta c$  for all  $c \in \bar{C} \subseteq U$  is strictly analogous. Finally,

$$\begin{aligned}
(\Delta^{\text{op}}h)\mathcal{R}_U &= (h_{(2)}f^a \otimes h_{(1)}\underline{S}e_a)\mathcal{R} = (h_{(2)}\triangleright f^a)h_{(3)} \otimes h_{(1)}\underline{S}e_a)\mathcal{R} \\
&= (f^ah_{(3)} \otimes h_{(1)}(\underline{S}e_a\triangleleft h_{(2)}))\mathcal{R} = (f^ah_{(2)} \otimes (\underline{S}e_a)h_{(1)})\mathcal{R} \\
&= (f^a \otimes \underline{S}e_a)\mathcal{R}(h_{(1)} \otimes h_{(2)}) = \mathcal{R}_U\Delta h
\end{aligned}$$

for all  $h \in H \subseteq U$ , using in order: the relations in  $\bar{C}\bowtie\bar{H} \subseteq U$ , the covariance of  $\text{exp}_B$ , the relation of  $H\bowtie B \subseteq U$ , and the quasicoassociativity axiom (1) for  $H$ . Finally, the element  $\mathcal{R}_U$  is manifestly invertible, with inverse  $\mathcal{R}^{-1}\text{exp}_B$ .  $\square$

We see that the quasitriangular structure of  $U$  is a product of the quasitriangular structure of  $H$  and the inverse of  $\exp_B$ . This is the reason that ‘q-exponentials’ of the root vectors appear in the quasitriangular structure of  $U_q(g)$ ; it is a rather general feature.

**Remark 3.7** Once we have constructed our algebra  $U$ , it is possible to use the relations of  $\bar{C} \bowtie \bar{H}$  and  $H \bowtie B$  to write the cross relations in Theorem 3.2 more compactly as

$$bc = \mathcal{R}^{(2)} c_{(\bar{2})} \underline{b}_{(\underline{2})} \mathcal{R}^{-(1)} \langle c_{(\bar{1})}, \underline{b}_{(\underline{1})} \triangleleft \mathcal{R}^{(1)} \rangle \langle \mathcal{R}^{-(2)} \triangleright \bar{S} c_{(\bar{3})}, \underline{b}_{(\underline{3})} \rangle \quad (24)$$

for all  $b \in B \subseteq U$  and  $c \in \bar{C} \subseteq U$ . Using the pairing relation (19) and axioms (1) we can also write these relations as

$$\underline{b}_{(\underline{1})} c_{(\bar{1})} \mathcal{R}^{-(1)} \langle \mathcal{R}^{-(2)} \triangleright c_{(\bar{2})}, \underline{b}_{(\underline{2})} \rangle = \langle c_{(\bar{1})}, \underline{b}_{(\underline{1})} \triangleleft \mathcal{R}^{(1)} \rangle \mathcal{R}^{(2)} c_{(\bar{2})} \underline{b}_{(\underline{2})}. \quad (25)$$

Both these forms are useful. Note however, that we have *not* defined  $U$  above as generated by  $\bar{C}, H, B$  and relations between them, but rather we have built  $U$  explicitly on the tensor product space  $\bar{C} \otimes H \otimes B$ . For this purpose, the form shown in Theorem 3.2 is more suited since it allows us to reorder  $bc$  into the canonical order in  $\bar{C} \otimes H \otimes B$ . Alternatively, one could take (24) etc. as defining cross relations of a Hopf algebra generated by  $U$ , but would then have to prove the ‘triangular decomposition’ that the product map  $\bar{C} \otimes H \otimes B$  is a linear isomorphism. This triangular decomposition is an intrinsic feature of our more explicit proofs above.

**Remark 3.8** We have written the formulae above in terms of  $\bar{C}$  rather a braided group  $C$  paired in the more categorical way to  $B$ . When  $H$  is a (quasitriangular) Hopf algebra, the two are entirely equivalent. Their underlying vector spaces and pairing maps with  $B$  can be identified, their products are opposite (in the usual sense) and their coproducts and  $H$ -module structures are related by

$$c_{(\bar{1})} \otimes c_{(\bar{2})} = c_{(\underline{1})} \triangleleft \mathcal{R}^{-(1)} \otimes c_{(\underline{2})} \triangleleft \mathcal{R}^{-(2)} \quad h \triangleright c = c \triangleleft Sh, \quad \forall h \in H. \quad (26)$$

This was explained at the start of the section. The explicit form uses the braiding in  ${}_H\mathcal{M}$  from (4) and the invariance of  $\mathcal{R}$  under  $S \otimes S$ . When viewed inside  $U \otimes U$ , we have the further identity

$$c_{(\bar{1})} \mathcal{R}^{-(1)} \otimes \mathcal{R}^{-(2)} \triangleright c_{(\bar{2})} = \mathcal{R}^{(1)} c_{(\underline{1})} \otimes c_{(\underline{2})} \triangleleft \mathcal{R}^{(2)}, \quad (27)$$

which follows from the relations of  $\bar{C} \bowtie \bar{H}$  and (1). Using such formulae, it is easy enough to rewrite all of the above in terms of  $C, H, B$  rather than in terms of  $\bar{C}, H, B$ . Explicitly, the structure of  $U$  in this form is

$$\begin{aligned} \Delta b &= b_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} b_{(2)}, & \Delta c &= \mathcal{R}^{(1)} c_{(1)} \otimes c_{(2)} \triangleleft \mathcal{R}^{(2)}, \\ bh &= h_{(1)}(b \triangleleft h_{(2)}), & ch &= h_{(2)}(c \triangleleft h_{(1)}) \\ b_{(1)} \mathcal{R}^{(1)} c_{(1)} \text{ev}(c_{(2)} \triangleleft \mathcal{R}^{(2)}, b_{(2)}) &= \text{ev}(c_{(1)}, b_{(1)} \triangleleft \mathcal{R}^{(1)}) c_{(2)} \mathcal{R}^{(2)} b_{(2)}. \end{aligned} \quad (28)$$

The *right-covariant* formulation with  $C$  has some advantages (and we will sometimes prefer it), because both  $B, C$  are then in the same braided category of right  $H$ -modules. The above *bicovariant* formulation in terms of  $\bar{C}$  (which is left-covariant) and  $B$  (which is right-covariant) is more natural from a purely algebraic point of view, allowing us to write  $U$  in a symmetrical way on  $\bar{C} \otimes H \otimes B$ .

In particular, we note that none of the above proofs of the quasitriangular bialgebra structure of  $U$  require an antipode for  $H$ ; we conclude that if  $H$  is a quasitriangular bialgebra,  $B$  a braided group in  $\mathcal{M}_H$ ,  $D \in {}_H\mathcal{M}$  is dually paired with  $B$  in the usual sense and  $\bar{C} = D^{\text{cop}}$ , then  $U$  defined in the same way as in Theorem 3.2 is a bialgebra. It is quasitriangular in the finite-type non-degenerately paired case.

**Remark 3.9** As another immediate generalisation, we note that none of the proofs of the Hopf algebra structure of  $U$  actually used  $\mathcal{R} \in H \otimes H$  itself but rather, in all expressions we find one component of  $\mathcal{R}$  or  $\mathcal{R}^{-1}$  acting on  $\bar{C}$  or  $B$ . Hence we are free to replace this combination by a coaction of a Hopf algebra  $A$  dual  $H$  and composition with one of the maps  $\mathcal{R}, \bar{\mathcal{R}} : A \rightarrow H$  or their convolution inverses. Hence we conclude that if  $(H, A)$  is weakly quasitriangular,  $B \in {}^A\mathcal{M}$  and  $C \in {}^A\mathcal{M}$  are categorically dually-paired (or  $D \in \mathcal{M}^A$  is ordinary-dually-paired) then there is a bialgebra or Hopf algebra  $U(\bar{C}, H, B)$  defined as above with the above changes. The relations of between these sub-Hopf algebras becomes

$$b_{(1)} c_{(\bar{1})} \bar{\mathcal{R}}(c_{(\bar{1})}^{(\bar{2})}) \langle c_{(\bar{2})}^{(\bar{1})}, b_{(2)} \rangle = \langle c_{(\bar{1})}, b_{(1)}^{(\bar{2})} \rangle \mathcal{R}(b_{(1)}^{(\bar{1})}) c_{(\bar{2})} b_{(2)} \quad (29)$$

and the coproduct is

$$\Delta b = b_{(1)}^{(\bar{2})} \otimes \mathcal{R}(b_{(1)}^{(\bar{1})}) b_{(2)}, \quad \Delta c = c_{(\bar{1})} \bar{\mathcal{R}}(c_{(\bar{2})}^{(\bar{2})}) \otimes c_{(\bar{2})}^{(\bar{1})}. \quad (30)$$

Note that if  $b, c$  are primitive elements (for their braided coproducts) then these relations simplify to

$$\begin{aligned} [b, c] &= \mathcal{R}(b^{(\bar{1})})\langle c, b^{(\bar{2})} \rangle - \langle c^{(\bar{1})}, b \rangle \bar{\mathcal{R}}(c^{(\bar{2})}) \\ \Delta b &= b^{(\bar{2})} \otimes \mathcal{R}(b^{(\bar{1})}) + 1 \otimes b, \quad \Delta c = c \otimes 1 + \bar{\mathcal{R}}(c^{(\bar{2})}) \otimes c^{(\bar{1})}. \end{aligned} \tag{31}$$

**Remark 3.10** Finally, we note that we also do not use directly the braided-antipodes  $B$  or  $C$  or their inverses in the proof of the bialgebra structure of  $U$ . What was actually used in the proofs was the composition  $\langle \bar{S}(\cdot), \cdot \rangle$  and its properties expressed in terms of the pairing in (18) and (19). Just as for the ordinary quantum double[22], one can extend the definitions to bialgebras by assuming in place of  $\langle \bar{S}(\cdot), \cdot \rangle$  a map  $\langle \cdot, \cdot \rangle^{-1}$  characterised as the inverse in the convolution algebra  $\text{hom}(D \otimes B, k)$  (the usual tensor product coalgebra here on  $D \otimes B$ ). In terms of  $\bar{C} \otimes B$ , this is equivalent to (18) and (19). Likewise for the proof of the quasitriangular structure, we need only assume the inverse of the coevaluation element  $\exp_B$  rather than the antipode of  $B$ .

These are all variations of the theory above; in our presentation we have chosen the most easily accessible framework (based on modules, assuming the antipodes etc.) for simplicity of presentation only, leaving the other cases as routine variations along established lines. We proceed now on the same basis for further general theory. To study the representation theory of  $U$ , however, we do need to break the left-right symmetry by working with either left  $U$ -(co)modules or right  $U$ -(co)modules. In the latter case (say) we work with  $B, C \in \mathcal{M}_H$  as explained in Remark 3.8.

**Lemma 3.11** *In the setting of Theorem 3.2, right  $U$ -module (algebras) can be identified with right  $H$ -module algebras  $V$  such that (i)  $V$  is braided  $B$ -module algebra in the braided category  $\mathcal{M}_H$  (ii) a left braided  $C$ -module algebra in the category  $\mathcal{M}_H$  (iii) the actions  $\triangleleft, \triangleright$  of  $B, C$  are compatible in the sense*

$$c_{(1)} \triangleright (v \triangleleft b_{(1)} \mathcal{R}^{(1)}) \text{ev}(c_{(2)} \triangleleft \mathcal{R}^{(2)} b_{(2)}) = \text{ev}(c_{(1)}, b_{(1)} \triangleleft \mathcal{R}^{(1)}) (c_{(2)} \triangleright v) \triangleleft \mathcal{R}^{(2)} b_{(2)}$$

for all  $v \in V$ ,  $b \in B \subseteq U$ ,  $c \in C \subseteq U$ .

**Proof** Since  $H \subseteq U$  is a sub-Hopf algebra, we require  $V$  an  $H$ -module algebra, i.e. an algebra in  $\mathcal{M}_H$ . By the bosonisation theorem[7, Thm 4.2] the modules of  $H \ltimes B$  and their tensor products coincide with the (right)  $B$ -modules in  $\mathcal{M}_H$  and their (braided) tensor products computed categorically in  $\mathcal{M}_H$ . This is part (i). For part (ii) the same picture applies for left  $\bar{C}$ -modules in  ${}_{\bar{H}}\mathcal{M}$ , but this is not directly applicable now. Instead, we need to realise the relations of  $\bar{C} \rtimes \bar{H}$  acting from the right. We view an action of  $\bar{C}$  from the right in the trivial way as an action of  $C$  from the left. For this to be a morphism in  $\mathcal{M}_H$ , we require  $(c \triangleright v) \triangleleft h = (c \triangleleft h_{(1)}) \triangleright (v \triangleleft h_{(2)})$ . In terms of the corresponding action of  $U$ , the left hand side is  $v \triangleleft ch$  and the right hand side is  $v \triangleleft (h_{(2)}(c \triangleleft h_{(1)}))$ , which is the cross relation in (28) for  $\bar{C} \rtimes \bar{H}$  when expressed in terms of  $C, H$ . The tensor product actions also match; thus  $c \triangleright (v \otimes w) = \underline{c_{(1)}} \triangleright (v \triangleleft \mathcal{R}^{(1)}) \otimes (\underline{c_{(2)}} \triangleleft \mathcal{R}^{(2)}) \triangleright w = v \triangleleft (\mathcal{R}^{-(1)} \underline{c_{(1)}}) \otimes w \triangleleft (\underline{c_{(2)}} \triangleleft \mathcal{R}^{-(2)}) = (v \otimes w) \triangleleft \Delta c$  which is the tensor product action according to the coproduct of  $U$ . The first equality here is the definition of the braided tensor product  $C$ -module[6] computed in the category  $\mathcal{M}_H$  using the braiding from (4). The last equality is (27). Finally, part (iii) is manifestly the requirement that these module-algebra structures in  $V$  respect the final cross relation in (28), which is equivalent to the reordering relation in Theorem 3.2.  $\square$

This digression into braided category theory is developed further in Appendix C. Here we give a purely algebraic consequence of it.

**Theorem 3.12** *In the setting of Theorem 3.2, the algebra  $V = B$  is a right  $U$ -module algebra by the maps*

$$v \triangleleft b = (\underline{S} \underline{b_{(1)}} \triangleleft \mathcal{R}^{(1)})(v \triangleleft \mathcal{R}^{(2)}) \underline{b_{(2)}}, \quad v \triangleleft c = \langle \bar{S} c, v_{(2)} \triangleleft \mathcal{R}^{-(2)} \rangle v_{(1)} \triangleleft \mathcal{R}^{-(1)}$$

*and the tautological action of  $H$  (i.e. the same as for  $B$  as an object in  $\mathcal{M}_H$ .) We call this the fundamental or Schrödinger representation of  $U$  on  $B$ .*

**Proof** This is best done diagrammatically; see the appendix Proposition C.4. The action of  $B$  is the right braided adjoint action of  $B$  on itself. The right action of  $\bar{C}$  is the left action of  $C$  from the proposition, which is the standard braided coregular representation. Both are known braided group constructions [3][15][14] and Proposition C.4 checks that they are suitably compatible to form a representation of  $U$ . We then convert over from  $C$  to  $\bar{C}$  as explained

in Remark 3.8. The restriction to  $\bar{C} \bowtie \bar{H}$  is a version of the fundamental representation used already in [13]. One can also verify directly by the same techniques as in the proofs above that these actions on  $V$  are compatible as required in part (iii) of the preceding lemma.  $\square$

Our previous Remarks 3.8 and 3.9 about only requiring  $H$  to be a bialgebra and only part of a weakly quasitriangular pair apply. The fundamental representation in Theorem 3.12 becomes in the weak case

$$v \triangleleft b = (\underline{S}b_{(1)}^{(\bar{2})})(v \triangleleft \mathcal{R}(b_{(1)}^{(\bar{1})}))b_{(2)}, \quad v \triangleleft c = \langle \bar{S}c, v_{(2)}^{(\bar{1})} \rangle v_{(1)} \triangleleft \bar{\mathcal{R}}(v_{(2)}^{(\bar{2})}). \quad (32)$$

If  $b$  is braided-primitive, we have

$$v \triangleleft b = vb - b^{(\bar{2})}(v \triangleleft \mathcal{R}(b^{(\bar{1})})), \quad (33)$$

which is the form that we will use. The action  $\triangleleft c$  is by braided differentiation[14] on the braided group  $B^{\text{cop}}$  with the braided opposite coproduct.

To conclude our general theory, we note that  $H \subseteq U$  as Hopf algebras. Hence by (a right handed version of) the transmutation theorem[6] there is a braided group  $\underline{U} = B(H, U) \in \mathcal{M}_H$  which consists of  $U$  as an algebra but has a modified coproduct. The action of  $H$  on  $\underline{U}$  is by the right adjoint action  $h \triangleleft g = (Sg_{(1)})hg_{(2)}$ , the given action on  $B$  and the action  $c \triangleleft g = (Sg_{(1)})cg_{(2)} = (Sg_{(2)} \triangleright c)(Sg_{(1)})g_{(3)}$  on  $\bar{C}$ . Here  $\underline{U} \supseteq B(H, H) \bowtie B$  a (right handed) braided group cross product by the braided group  $B(H, H)$  associated to the identity mapping. Indeed, this is the original construction of the bosonisation  $H \bowtie B$  in [7] as such that its transmutation is a cross product. Moreover, in the finite-dimensional non-degenerately paired case we know from [6] that  $\underline{U}$  is braided-quasitriangular (a quantum braided group in the strict sense). Explicitly, the braided coproducts on  $H, B \subseteq \underline{U}$  and its braided-quasitriangular structure are

$$\Delta_{\underline{U}}h = h_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes (S\mathcal{R}^{(2)})h_{(2)}, \quad \Delta_{\underline{U}}b = b_{(1)} \triangleleft \mathcal{R}^{(1)} \otimes (S\mathcal{R}^{(2)})b_{(2)}, \quad \mathcal{R}_{\underline{U}} = \sum f^a \triangleleft \mathcal{R}^{(1)} \otimes (S\mathcal{R}^{(2)})\underline{S}e_a, \quad (34)$$

while  $\Delta_{\underline{U}}$  restricted to  $\bar{C}$  is more complicated. We can also transmute by  $\bar{H} \subseteq U$ , in which case the restriction to  $\bar{C}$  is simple and the restriction to  $B$  more complicated.



## 4 Recovering Lusztig's construction of $U_q(g)$

A *Cartan datum* in Lusztig's construction of  $U_q(g)$  is a set  $I = \{i\}$  and a symmetric bilinear form  $\cdot$  on  $\mathbb{Z}[I]$  such that

$$i \cdot i \in \{2, 4, 6, \dots\}, \quad a_{ij} \equiv \frac{2i \cdot j}{i \cdot i} \in \{0, -1, -2, \dots\}, \quad \forall i \neq j. \quad (35)$$

Thus,  $a_{ij}$  is a symmetrizable Cartan matrix. Also defined is a *root datum*, which is two finitely generated free Abelian groups  $Y, X$  with a perfect pairing  $\langle \cdot, \cdot \rangle : Y \times X \rightarrow \mathbb{Z}$  and inclusions (the second denoted  $i \mapsto i'$ )

$$Y \supseteq I \subseteq X, \quad \text{s.t.} \quad \langle i, j' \rangle = a_{ij}. \quad (36)$$

We show now that these data provide now a weakly quasitriangular dual pair in the sense explained in the Preliminaries. We let  $H = kY$  with basis elements  $\{K_\mu : \mu \in Y\}$ . This forms a Hopf algebra with  $\Delta K_\mu = K_\mu \otimes K_\mu$  and product  $K_\mu K_\nu = K_{\mu+\nu}$ , extended linearly. Let  $A = k\mathbb{Z}[I] = k[g_i, g_i^{-1}]$  the group algebra of  $\mathbb{Z}[I]$  with  $\Delta g_i = g_i \otimes g_i$ . Using the Cartan datum (symmetric or not) it is clear that  $A$  is dual quasitriangular, with  $\mathcal{R}(g_i, g_j) = q^{i \cdot j}$ , for any  $q \in k^*$ . This more general point of view is relevant to Appendix B. For the present we really need a weakly quasitriangular dual pair. We can work over  $k = \mathbb{Q}(q)$ , for example.

**Lemma 4.1** *Using the root datum, we define a pairing*

$$\langle \cdot, \cdot \rangle : H \otimes A \rightarrow k, \quad \langle K_\mu, g_i \rangle = q^{\langle \mu, i' \rangle} \quad (37)$$

*and relative to it, we have a weak quasitriangular structure*

$$\mathcal{R}, \bar{\mathcal{R}} : A \rightarrow H, \quad \mathcal{R}(g_i) = K_i^{\frac{i \cdot i}{2}}, \quad \bar{\mathcal{R}}(g_i) = K_i^{-\frac{i \cdot i}{2}}. \quad (38)$$

**Proof** The inclusion  $I \subseteq X$  induces a homomorphism  $\mathbb{Z}[I] \rightarrow X$  and hence a homomorphism  $A \rightarrow kX$  of Hopf algebras. The latter is dually paired with  $kY$  via the assumed group pairing. This gives the pairing between  $H, A$  (it need no longer be non-degenerate, however). We also have well-defined algebra homomorphisms  $\mathcal{R}, \bar{\mathcal{R}}$  as stated. Here  $\mathcal{R}^{-1} = \bar{\mathcal{R}}$  as a consequence of the symmetry of  $\cdot$ . We check that

$$\langle \mathcal{R}^{-1}(g_j), g_i \rangle = q^{-\frac{i \cdot j}{2} \langle j, i' \rangle} = q^{-j \cdot i} = q^{-i \cdot j} = q^{-\frac{i \cdot i}{2} \langle i, j' \rangle} = \langle K_i^{-\frac{i \cdot i}{2}}, g_j \rangle = \langle \bar{\mathcal{R}}(g_i), g_j \rangle$$

in virtue of the symmetry.  $\square$

Next, we let  $\tilde{B} = k\langle e^i \rangle$  the free non-commutative algebra on  $I$ . This lives in the braided category of left  $A$ -comodules by  $e^i \mapsto g_i \otimes e^i$ . Hence it also lives in the category of right  $H$ -modules by

$$e^i \triangleleft K_\mu = \langle K_\mu, g_i \rangle e^i = q^{\langle \mu, i' \rangle} e^i. \quad (39)$$

**Lemma 4.2** *The maps defined by  $\underline{\Delta}e^i = e^i \otimes 1 + 1 \otimes e^i$ ,  $\underline{\epsilon}e^i = 0$ ,  $\underline{S}e^i = -e^i$  make  $\tilde{B}$  a braided group in the category of left  $A$ -comodules with braiding provided by the weak quasitriangular structure from Lemma 4.1.*

**Proof** We check that the braiding as defined by the weak quasitriangular structure is the desired one, namely

$$\Psi(e^i \otimes e^j) = e^j \otimes e^i \triangleleft \mathcal{R}(g_j) = e^j \otimes e^i \triangleleft K_j^{\frac{i \cdot j}{2}} = e^j \otimes e^i q^{\langle j, i' \rangle \frac{i \cdot j}{2}} = q^{j \cdot i} e^j \otimes e^i.$$

The rest is clear from [28][13][14] or the computations in Lusztig [10]. Indeed, the theory of braided groups ensures that we have natural braided tensor product algebras  $\tilde{B} \underline{\otimes} \tilde{B}$  (and higher braided tensor products as well). The relations are

$$(1 \otimes e^i)(e^j \otimes 1) = q^{j \cdot i}(e^j \otimes 1)(1 \otimes e^i).$$

We extend  $\underline{\Delta} : \tilde{B} \rightarrow \tilde{B} \underline{\otimes} \tilde{B}$  as an algebra homomorphism, and  $\underline{S} : \tilde{B} \rightarrow \tilde{B}$  as a braided anti-homomorphism using  $\Psi$ .  $\square$

It is also clear from [14] or computations in [10] that  $\tilde{B}$  has ordinary-dual  $\tilde{D} = k\langle f_i \rangle$  in the category of right  $A$ -comodules by  $f_i \mapsto f_i \otimes g_i$ . It also lives in the category of left  $H$ -modules by  $K_\mu \triangleright f_i = \langle K_\mu, g_i \rangle f_i = q^{\langle \mu, i' \rangle} f_i$  and forms a braided group with  $f_i$  braided-primitive and  $\Psi(f_i \otimes f_j) = q^{i \cdot j} f_j \otimes f_i$ . We take the pairing with  $\tilde{B}$  to be

$$\langle f_i, e^j \rangle = (q_i - q_i^{-1})^{-1} \delta_i^j, \quad (40)$$

where  $q_i = q^{\frac{i \cdot i}{2}}$ ; we inserted here a choice of normalisation factor for each  $e^i$ . Following Lusztig, we pass to the quotients  $B, D$  by the radical of the pairing, generated by the  $q$ -Serre-relations. We let  $\bar{C} = D^{\text{cop}}$ .

**Proposition 4.3** *The  $U(\bar{C}, H, B)$  construction in Section 3 reduces in this setting to Lusztig's construction of  $U_q(g)$  in suitable (right-handed) conventions.*

**Proof** The braided group  $\bar{C}$  has the primitive braided coproduct  $\bar{\Delta}f_i = f_i \otimes 1 + 1 \otimes f_i$  (because  $\Psi(1 \otimes f_i) = f_i \otimes 1$  etc.) But it extends to products with the inverse braiding to that of  $D[4]$ , which in our case means

$$\Psi(f_i \otimes f_j) = \bar{\mathcal{R}}(g_i) \triangleright f_j \otimes f_i = q^{-i \cdot j} f_j \otimes f_i.$$

The algebras  $\bar{C}, H, B$  are all included in  $U$ . From the relations  $bh = h_{(1)}(b \triangleleft h_{(2)})$  and  $hc = (h_{(1)} \triangleright c)h_{(2)}$  in  $H \bowtie B$  and  $\bar{C} \bowtie \bar{H}$ , we have

$$e^i K_\mu = \langle K_\mu, g_i \rangle K_\mu e^i = q^{\langle \mu, i' \rangle} K_\mu e^i, \quad K_\mu f_i = f_i K_\mu \langle K_\mu, g_i \rangle = q^{\langle \mu, i' \rangle} f_i K_\mu.$$

From the formulae (31) we have the cross relations and coproducts

$$\begin{aligned} [e^i, f_j] &= (\mathcal{R}(g_i) - \bar{\mathcal{R}}(g_j)) \langle f_j, e^i \rangle = \frac{K_i^{\frac{i \cdot j}{2}} - K_i^{-\frac{i \cdot j}{2}}}{q_i - q_i^{-1}} \delta^i_j \\ \Delta e^i &= e^i \otimes K_i^{\frac{i \cdot i}{2}} + 1 \otimes e^i, \quad \Delta f_i = f_i \otimes 1 + K_i^{-\frac{i \cdot i}{2}} \otimes f_i. \end{aligned}$$

Hence we recover the structure of  $U_q(g)$  in [10] in suitable conventions. The identification is by  $e^i = -E_i$ ,  $f_i = F_i$  and an interchange of  $K_i$  with  $K_i^{-1}$ .  $\square$

The relations between non-primitive elements of  $\bar{C}, B$  and their coproducts follow just as easily from Theorem 3.2, as

$$bc = \tilde{K}_{|b_{(1)}|} c_{(\bar{2})} b_{(1)} \tilde{K}_{|c_{(\bar{3})}|}^{-1} \langle c_{(\bar{1})}, b_{(1)} \rangle \langle \bar{S} c_{(\bar{3})}, b_{(\bar{3})} \rangle, \quad \Delta b = b_{(1)} \otimes \tilde{K}_{|b_{(1)}|} b_{(2)}, \quad \Delta c = c_{(\bar{1})} \tilde{K}_{|c_{(\bar{2})}|}^{-1} \otimes c_{(\bar{2})} \quad (41)$$

for all  $b \in B$ ,  $c \in \bar{C}$  of homogeneous degree. Here  $|e^i| = |f_i| = i \in \mathbb{Z}[I]$  corresponding to the coactions above, and  $\tilde{K}_{\sum_i \nu_i i} \equiv \prod_i K_i^{\frac{i \cdot i}{2} \nu_i} = \mathcal{R}(\prod_i g_i^{\nu_i})$  for  $\sum_i \nu_i i \in \mathbb{Z}[I]$ . We also deduce the triangular decomposition of  $U_q(g)$  into  $\bar{C}, H, B$ . These facts and formulae all require substantial proof in [10, Sec. 3.1.5, Prop. 3.1.7, Sec. 3.2], where  $U_q(g)$  is defined by generators and relations.

We are also in a position to apply general constructions for braided groups, e.g. proven diagrammatically, to obtain results about  $U_q(g)$  somewhat more easily than by the usual direct calculation.

**Proposition 4.4** *The fundamental representation in Theorem 3.12 of  $U$  on  $B$ , with generators denoted  $x^i$ , is*

$$x^i \triangleleft K_\mu = q^{\langle \mu, i' \rangle} x^i, \quad v(x) \triangleleft e^i = \frac{v(x)x^i - x^i v(x \triangleleft K_i^{\frac{i \cdot i}{2}})}{q_i - q_i^{-1}}, \quad v(x) \triangleleft f_i = -\partial_i v(x),$$

where  $\partial_i$  is the braided differentiation

$$\partial_i \left( (x^1)^{\nu_1} \dots (x^r)^{\nu_r} \right) = q^{-i \cdot \sum_{j=1}^{i-1} \nu_j j} (x^1)^{\nu_1} \dots (x^{i-1})^{\nu_{i-1}} [\nu_i, q_i^{-2}] (x^i)^{\nu_i-1} (x^{i+1})^{\nu_{i+1}} \dots (x^r)^{\nu_r}$$

where  $I = \{1, \dots, r\}$  and  $[m, q] = \frac{1-q^m}{1-q}$ . This representation is adjoint to a version of Lusztig's Verma module representation in [10]. Moreover, it makes  $B$  into a  $U$ -module algebra.

**Proof** The action of  $H$  is the given action on  $B$  from the right. The action of  $e^i$  is the right braided adjoint action (33). More precisely, we have rescaled the  $e^i$  in (40) by a factor  $(q_i - q_i^{-1})^{-1}$ , whereas we keep the  $x^i$  in the usual normalisation where  $\langle f_i, x^j \rangle = \delta_i^j$ . For the action of  $f_i$ , we write the action in (33) as

$$v \triangleleft f_i = \langle \bar{S} f_i, v_{(1)}^{\text{op}} \rangle v_{(2)}^{\text{op}} = -\partial_i v$$

where  $v_{(1)}^{\text{op}} \otimes v_{(2)}^{\text{op}} \equiv \Psi^{-1}(v_{(1)} \otimes v_{(2)})$  is the coproduct of  $B^{\text{cop}}$ . This has the same linear form on the generators  $x^i$  since  $\Psi(x^i \otimes 1) = 1 \otimes x^i$  etc., but extends to products with the opposite braiding according to [4]. Hence we can compute this as shown, where  $\partial_i v$  is characterised by  $v_{(1)}^{\text{op}} \otimes v_{(2)} = x^i \otimes \partial_i v + \text{terms not of the form } x^i \otimes (\ )$ . It is a version of the operator  ${}_i r$  in [10]. The braided-module algebra properties of the braided-coregular representation correspond to the braided-Leibniz rule[5][14], which in the present case takes the form  $\partial_i(vw) = (\partial_i v)w + q^{-i \cdot |v|} v \partial_i w$  for  $v, w \in B$  and  $v$  homogeneous, as in [5][10]. From this, we find easily the explicit formula shown for  $\partial_i$ . It can also be obtained from (11) with  $R_j^i k_l = \delta_j^i k_l q^{-i \cdot k}$ .

By contrast, the Verma representation in [10, Sec. 3.4.5] consists of  $f_i$  acting by multiplication on  $\bar{C}$  (both  $B$  and  $\bar{C}$  are versions of Lusztig's algebra  $f$ ). The action of  $e^i$  is a difference of the differentiation operators  ${}_i r$  and  $r_i$  in [10]. This is clearly the adjoint of the fundamental representation above: the adjoint under the duality pairing of braided differentiation is braided multiplication. Note, however, that  $U$  does not respect the algebra structure of  $\bar{C}$  (it respects its coalgebra). For this reason we consider the action on  $B$  rather than its adjoint to be more fundamental, since by Theorem 3.12 we know that  $V = B$  becomes a  $U$ -module algebra.  $\square$

There is also a weak quasitriangular structure which can be obtained from a version of Proposition 3.6, at least for the classical ABCD series of finite-dimensional semisimple Lie algebras. It requires, however, a description of the coordinate algebra  $G_q$  dual to  $U_q(g)$ , which we do not cover here. We remark only that in a suitable setting where we work over formal power series[11], we obtain the formula

$$\mathcal{R}_{U_q(g)} = \text{exp}_B \mathcal{R}_H; \quad \mathcal{R}_H = q^{\sum_{i,j} h_i \otimes h_j \frac{i \cdot j}{2} (a^{-1})_{ij}}.$$

Here we assume  $K_i = q^{h_i}$  and  $\text{exp}_B$  is obtained from the duality pairing coevaluation. Because the duality pairing is non-degenerate we know that  $\text{exp}_B$  exists as a formal power series built from a basis and dual basis of  $B$ . It corresponds to the element  $\Theta$  studied in [10, Sec. 4].

In the simplest case we take for  $B$  the braided line  $k\langle e \rangle$  and for  $\bar{C}$  the braided line  $k\langle f \rangle$ . Then  $U = U_q(sl_2)$ . Its fundamental representation on  $k\langle x \rangle$  is

$$v(x) \triangleleft K = v(q^2 x), \quad v \triangleleft e = -qx^2 \partial_{q^2} v, \quad v \triangleleft f = -\partial_{q^{-2}} v, \quad (42)$$

where  $\partial_q v(x) = \frac{v(x) - v(qx)}{(1-q)x}$  is the usual 1-dimensional  $q$ -derivative on polynomials  $v \in k\langle x \rangle$ . The fundamental representation in this case is a  $q$ -deformation of the action on  $k\langle x \rangle$  of  $sl_2$  as the degree 1,0,-1 subalgebra of the Virasoro algebra in physics. Meanwhile, the braided exponential part of the quasitriangular structure comes out as

$$\text{exp}_B = e_{q^{-2}}^{-(q-q^{-1})f \otimes e}, \quad (43)$$

as in [29], where  $e_q$  denotes the usual  $q$ -exponential defined with  $[m; q]! = [m; q] \cdots [2; q]$  in place of  $m!$ .

This completes our outline of how the explicit constructions in [10, Part I] and other results can be recovered as an application of the double bosonisation and its properties in Section 3. The theory applies in other related settings just as well:

**Example 4.5** *If  $q \neq 1$  is a primitive  $r$ -th root of 1 with  $r$  odd and invertible in  $k$ , we take for  $B$  the anyonic line  $k\langle e \rangle / e^r$  from [28]. This forms a braided group in the category of  $k\mathbb{Z}/r\mathbb{Z}$ -comodules (or modules, since  $\mathbb{Z}/r\mathbb{Z}$  is self-dual). Then  $U(B)$  is the finite-dimensional reduced form  $u_q(sl_2)$  with  $K^r = 1, e^r = 0 = f^r$ . The fundamental representation on  $k\langle x \rangle / x^r$  is as in*

(42). The quasitriangular structure from Proposition 3.6 is the product of

$$\text{exp}_B = \sum_{m=0}^{r-1} f^m \otimes (-e)^m (q - q^{-1})^m ([m; q^{-2}]!)^{-1}, \quad \mathcal{R}_H = r^{-1} \sum_{m,n=0}^{r-1} q^{-2mn} K^m \otimes K^n.$$

**Proof** We use the quasitriangular  $\mathcal{R}_H$  structure on  $H = k\mathbb{Z}/r\mathbb{Z}$  introduced in [28]. The braided group  $B = k\langle e \rangle / e^r$  is also introduced there, with  $|e| = 1$  and  $\underline{\Delta}e = e \otimes 1 + 1 \otimes e$ . We take a slightly different braiding (to fit with the conventions above), namely  $\Psi(e^m \otimes e^n) = q^{2mn} e^n \otimes e^m$  defined with  $q^2$  in place of  $q$ . Similarly for  $k\langle f \rangle / f^r$ . The pairing  $\langle f^m, e^n \rangle = \delta_n^m [m; q^2]!$  is non-degenerate between these finite-dimensional braided groups. Hence the coevaluation is  $\text{exp}_B = \sum_{m=0}^{r-1} e^m \otimes f^m (q - q^{-1})^m ([m; q^2]!)^{-1}$ . The braided-antimultiplicativity of the braided antipode implies that  $\underline{S}(e^m) = q^{m(m-1)} (-e)^m$ , which gives  $\text{exp}_B$  as stated in the example (the same applies to the formal power series (43)). We also check that the quasitriangular structure  $\mathcal{R}_H$  induces the correct weak quasitriangular structure, by evaluation. Thus  $A = k\mathbb{Z}/r\mathbb{Z}$  with pairing  $\langle K, g \rangle = q^2$  and  $\langle \mathcal{R}^{(1)}, g \rangle \mathcal{R}^{(2)} = r^{-1} \sum_{m,l=0}^{r-1} q^{-2ml} q^{2m} K^l = K$  since  $r^{-1} \sum_{m=0}^{r-1} q^{m(2(n-1))} = \delta_{2(n-1),0} = \delta_{n,1}$ . The Kronecker delta functions here are on  $\mathbb{Z}/r\mathbb{Z}$ .  $\square$

This recovers  $u_q(sl_2)$  as in [30] and (with the quasitriangular structure) [28][27]. The braided version  $\underline{U}$  from (34) recovers the anyonic quantum group  $\underline{u}_q(sl_2)$  in [28]. In a different direction, we can include as well the case where  $\cdot$  on  $\mathbb{Z}[I]$  is antisymmetric. For example, we can suppose  $\langle i, j' \rangle = i \cdot j$  and define  $\mathcal{R}(g_i) = K_i = \bar{\mathcal{R}}(g_i)$  as a weak triangular structure. This case is not very interesting for us, however: The category of comodules in this case is symmetric rather than braided and the relations in (31) reduce to  $[b, c] = 0$  for all braided-primitive  $b \in B$  and  $c \in \bar{C}$ .

## 5 New quantum group constructions

We now apply our constructions to more general quantum groups  $H$  in the role of ‘Cartan’ subalgebra. Here the main datum we need is a  $R$ -matrix, i.e. a matrix  $R \in M_n \otimes M_n$  which is an (invertible) solution of the matrix Yang-Baxter equations. We let  $A(R)$  denote the usual matrix bialgebra with generators  $\mathbf{t} = \{t^i_j\}$ , relations  $R\mathbf{t}_1\mathbf{t}_2 = \mathbf{t}_2\mathbf{t}_1R$  and coproduct  $\Delta\mathbf{t} = \mathbf{t} \otimes \mathbf{t}$  in standard notation[31]. We let  $\widetilde{U(R)} = A(R) \bowtie A(R)$  the double cross product bialgebra constructed in [22]. It consists of two copies of  $A(R)$  with generators  $\mathbf{m}^-, \mathbf{m}^+$ , say, and the

cross relations and coalgebra

$$R\mathbf{m}_1^+\mathbf{m}_2^- = \mathbf{m}_2^-\mathbf{m}_1^+R, \quad \Delta m^{\pm i}_j = \sum_a m^{\pm a}_j \otimes m^{\pm i}_a, \quad \epsilon m^{\pm i}_j = \delta^i_j. \quad (44)$$

Combining results in [22] for the weak quasitriangular structure and [5] for the braided plane  $V(R)$ , we have:

**Lemma 5.1** *Let  $R$  be an invertible matrix solution of the QYBE. Then  $(\widetilde{U(R)}, A(R))$  is a weakly quasitriangular dual pair with  $\mathcal{R}(\mathbf{t}) = \mathbf{m}^+$  and  $\bar{\mathcal{R}}(\mathbf{t}) = \mathbf{m}^-$  and  $V(R) = k\langle e^i \rangle$  is a braided group with  $\underline{\Delta}e^i = e^i \otimes 1 + 1 \otimes e^i$  extended by the corresponding braiding.*

**Proof** Motivated by [31], we considered in [22] the bialgebras  $A(R)$  and  $\widetilde{U(R)}$  as dually paired by  $\langle m^{+i}_j, t^k_l \rangle = R^i_j{}^k_l$  and  $\langle m^{-i}_j, t^k_l \rangle = R^{-1k}_l{}^i_j$ , and verified that  $\mathcal{R}$  extends as a weak quasitriangular structure. The same applies for  $\bar{\mathcal{R}}$ , and the two are clearly inverse-transpose as required for a weak quasitriangular structure in the sense (3). There is left coaction of  $A(R)$  on  $V(R)$  given by  $e^i \mapsto t^i_a \otimes e^a$ , which induces a right action of  $\widetilde{U(R)}$  by

$$e^i \triangleleft m^{+j}_k = e^a \langle m^{+j}_k, t^i_a \rangle = R^j_k{}^i_a e^a, \quad e^i \triangleleft m^{-j}_k = e^a \langle m^{-j}_k, t^i_a \rangle = R^{-1i}_a{}^j_k e^a. \quad (45)$$

The induced braiding  $\Psi(e^i \otimes e^j) = e^a \otimes e^i \triangleleft \mathcal{R}(t^j_a)$  is then the correct one for  $V(R)$  as required in (9) in the Preliminaries. We use it to define a braided tensor product algebra  $V(R) \underline{\otimes} V(R)$  and extend  $\underline{\Delta}$  as an algebra homomorphism to it[5].  $\square$

We take this for  $B$  and we take  $D = V^\sim(R) = k\langle f_i \rangle$  as described in the Preliminaries. The braided group  $\bar{C}$  has the same algebra as  $D$  and the same linear form of the braided-coproduct on the generators  $f_i$ , but extended to products with the inverse braiding, i.e. it is  $V^\sim(R_{21}^{-1})$ . The bialgebra version of Theorem 3.2 yields:

**Proposition 5.2** *There is a bialgebra  $U = U(V(R), \widetilde{U(R)}, V^\sim(R_{21}^{-1}))$  generated by  $\mathbf{m}^\pm, \mathbf{e} = \{e^i\}, \mathbf{f} = \{f_i\}$  with the cross relations and coproduct*

$$\begin{aligned} e^i m^{+j}_k &= R^j_a{}^i_b m^{+a}_k e^b, \quad m^{-i}_j e^k = R^k_a{}^i_b e^a m^{-b}_j, \\ m^{+i}_j f_k &= f_b m^{+i}_a R^a_j{}^b_k, \quad f_i m^{-j}_k = m^{-j}_b f_a R^a_i{}^b_k, \quad [e^i, f_j] = \frac{m^{+i}_j - m^{-i}_j}{q - q^{-1}} \\ \Delta e^i &= e^a \otimes m^{+i}_a + 1 \otimes e^a, \quad \Delta f_i = f_i \otimes 1 + m^{-a}_i \otimes f_a, \quad \epsilon e^i = \epsilon f_i = 0. \end{aligned}$$

Here  $\widetilde{U(R)}$  appears as a sub-bialgebra. The factor  $q - q^{-1} \in k^*$  is an arbitrary choice of normalisation for the  $e^i$ , chosen for conventional purposes.

**Proof** The action (45) and a similar computation for the action on  $f_i$ , immediately give the relations with  $\mathbf{m}^\pm$  (using the matrix form of the coproduct of the latter). We then use the formulae for the pairing and weak quasitriangular structure to compute the cross relations and coproduct from (31), giving the results stated. In doing so, we introduce an overall normalisation factor  $(q - q^{-1})$  for the  $e^i$ , so that  $\langle f_i, e^j \rangle = (q - q^{-1})^{-1} \delta_i^j$  for some  $q$  with  $q^2 \neq 0, 1$ . This is purely conventional to suit the examples of interest; it can be any constant.  $\square$

At this level, the formulae (11)–(14) from [14] provide us with the structure needed for  $\text{exp}_B$  and for the fundamental representation. Thus,

**Proposition 5.3** *In the setting of Proposition 5.2, the free algebra  $V(R)$ , generated by  $x^i$ , is a  $U$ -module algebra by*

$$x^i \triangleleft m^{+j}_k = R^j_k{}^i{}_a x^a, \quad x^i \triangleleft m^{-j}_k = R^{-1i}{}_a{}^j{}_k x^a, \quad v \triangleleft e^i = \frac{vx^i - x^a(v \triangleleft m^{+i}_a)}{q - q^{-1}}, \quad v \triangleleft f_i = -\partial_i v$$

for all  $v \in V(R)$ , where  $\partial_i$  is the braided differentiation [14] on  $V(R_{21}^{-1})$  from (11).

**Proof** We action of  $\mathbf{m}^\pm$  is the given action on  $B$  in (45). We use the right braided adjoint action (33), computed again from Lemma 5.1, for the action of  $e^i$ . We take  $x^i$  in their natural normalisation where the pairing with  $f_i$  is  $\langle f_i, x^j \rangle = \delta_i^j$ . For the action of  $f_i$  we use the formulation of the coregular representation as braided-differentiation introduced in [14]; as in the proof of Proposition 4.4, we write it as evaluation against the coproduct of  $V(R)^{\text{cop}}$ , which is  $V(R)$  with the opposite braiding, i.e.  $V(R_{21}^{-1})$ . Hence  $\partial_i$  is given on monomials by the braided-integer matrices  $[m; R_{21}^{-1}]$  in the notation of the Preliminaries.  $\square$

Next we consider the same construction at the level of quotient Hopf algebras and quotient braided groups. These steps depend in fact on the normalisation of  $R$ . In the framework of [5] where braided groups with quadratic relations are constructed from  $k\langle e^i \rangle$ , a necessary condition is that the matrix  $PR$  has an eigenvalue  $-1$ . The possible quadratic relations are determined by a matrix  $R'$  as in (15), obeying certain conditions. We fix a choice of  $R, R'$  (the braided plane data). Given these, we look for quotients of the bialgebras  $\widetilde{U(R)}$  and  $A(R)$  which are Hopf algebras, such that the pairing and weak quasitriangular structure descend. Typically, this is possible provided we normalise  $R$  (which enters into the pairing) correctly, e.g. provided we



modify the pairing to  $\langle \mathbf{m}_1^+, \mathbf{t}_2 \rangle = \lambda R$  and  $\langle \mathbf{m}_1^-, \mathbf{t}_2 \rangle = \lambda R_{21}^{-1}$ , where  $\lambda \in k^*$  is a suitable constant. We say that such  $R$  is *regular* and that  $\lambda$  is a *quantum group normalisation constant*. This framework has been introduced (in an equivalent form) in [5].

The  $R$ -matrices for the standard  $ABCD$  series of Lie algebras are known [31] and are regular in this sense when we work over  $\mathbb{C}$ , with quotient weakly quasitriangular dual pair  $(U_q(g), G_q)$  in suitable form (which may be slightly different, however, from Lusztig's ‘minimal’ form in Section 4). Then we identify the image  $\mathbf{m}^\pm = S\mathbf{1}^\mp$  in the conventional notation of [31], where  $S$  is the antipode of  $U_q(g)$ . Other examples of interest include the  $q$ -Lorentz group dual pair [32]. The quantum-braided planes  $V(R)$  and  $V^\sim(R)$  and their suitable quotients remain covariant under the quotients  $G_q$  etc. of  $A(R)$ . Lemma 5.1 no longer goes through, however.

**Lemma 5.4** *cf. [5] Let  $R$  be regular and  $(H, A)$  a quantum group quotient of  $(\widetilde{U(R)}, A(R))$  with associated normalisation constant  $\lambda$ . Let  $\tilde{H} = H \otimes k[c]$  and  $\tilde{A} = A \otimes k[g]$  be the centrally extended weakly quasitriangular dual pair defined by*

$$\Delta c = c \otimes c, \quad \Delta g = g \otimes g, \quad \langle c, g \rangle = \lambda, \quad \mathcal{R}(g) = c^{-1}, \quad \bar{\mathcal{R}}(g) = c.$$

*Then  $V(R)$  and  $V^\sim(R)$  (and their covariant quotients) are braided groups in the category of  $\tilde{A}$ -comodules by the coactions  $e^i \mapsto gt^i_a \otimes e^a$  and  $f_i \mapsto f_a \otimes gt^a_i$ .*

**Proof** We cast the construction in [5] into the present weakly quasitriangular setting. Indeed, the induced action of  $c$  is  $e^i \triangleleft c = \lambda e^i$ , hence  $\Psi(e^i \otimes e^j) = e^a \otimes e^i \triangleleft \mathcal{R}(gt^j_a) = e^a \otimes e^i \triangleleft c^{-1} m^{+j}_a = R^j_a{}^i e^a \otimes e^b$  as required. Similarly  $f_i \triangleleft c = \lambda f_i$  adjusts correctly for the braiding.  $\square$

We can then make the bosonisations  $\tilde{H} \bowtie V(R', R)$  etc., which is the general construction of inhomogeneous quantum introduced in [5]. The elements  $c, g$  are called in this context ‘dilaton generators’. The construction in [5] recovered some of the specific examples of inhomogeneous quantum groups obtained by other means. We make the same extension when constructing our Hopf algebra  $U$ , whenever the appropriate quantum group normalisation constant is not 1. To be concrete, we specialise to one of the standard weakly quasitriangular dual pairs  $(U_q(g), G_q)$  and  $\mathbb{A}_q = V(R', R)$  a choice of quantum-braided plane covariant as algebras under  $G_q$  (i.e. of  $G_q$  type) and  $\bar{\mathbb{A}}^\star_q = V^\sim(R', R_{21}^{-1})$  its dual with braided-opposite coproduct. The same formulae hold at the level of generality of Lemma 5.4.

**Corollary 5.5** *Let  $\mathbb{A}_q$  be a quantum-braided plane of  $G_q$  type and  $\lambda$  the associated quantum group normalisation constant. Then  $U = U(\mathbb{A}_q, \widetilde{U_q(g)}, \bar{\mathbb{A}}_q^*)$  has the  $m^\pm$  relations as in Proposition 5.2 with  $\lambda R$  in place of  $R$ , and the cross relations and coalgebra*

$$\begin{aligned} cf_i &= \lambda f_i c, & e^i c &= \lambda c e^i, & [c, \mathbf{m}^\pm] &= 0, & [e^i, f_j] &= \frac{m^{+i}_j c^{-1} - c m^{-i}_j}{q - q^{-1}} \\ \Delta c &= c \otimes c, & \Delta e^i &= e^a \otimes m^{+i}_a c^{-1} + 1 \otimes e^a, & \Delta f_i &= f_i \otimes 1 + c m^{-a}_i \otimes f_a. \end{aligned}$$

**Proof** We repeat the computations from (31), using this time the coactions and weak quasitriangular structure of  $(\widetilde{U_q(g)}, \widetilde{G_q})$  from Lemma 5.2. Because  $\mathbb{A}_q$  and  $\bar{\mathbb{A}}_q^*$  are well-defined braided groups in the corresponding braided category, we know that the our previous calculations can be made at this level.  $\square$

The fundamental representation of  $U$  on  $\mathbb{A}_q$  also descends to this quotient level. The formulae in Proposition 5.3 become

$$v(x) \triangleleft c = v(\lambda x), \quad v \triangleleft e^i = \frac{v x^i - x^a (v \triangleleft c^{-1} m^{+i}_a)}{q - q^{-1}}, \quad v \triangleleft f_i = -\partial_i v \quad (46)$$

and  $\lambda R$  in place of  $R$  for the action of  $\mathbf{m}^\pm$ . That  $\partial_i$  descend to the quotients (15) is shown in [14]. The first and last actions in (46) provide the fundamental representation of the  $q$ -Poincaré algebra in  $q$ -spacetime as a module algebra c.f.[5][8], for the appropriate regular  $R$ -matrix data and quotients. Our double bosonisation construction extends this approach to the  $q$ -conformal group defined by  $U$ , with the  $e^i$  the additional generators acting as in (46). This geometrical picture of (46) will be developed elsewhere.

It should be clear that Corollary 5.5 leads to new quantum groups even when  $U_q(g)$  is one of the standard  $q$ -deformations of ABCD type. For in these cases there is more than one possible choice of quantum plane  $\mathbb{A}_q$ . For the A series, there are two choices, namely of ‘fermionic’ or ‘bosonic’ type. For generic  $q$  the latter has the same dimensions at each degree as the classical polynomial algebra in  $n$  variables. There is such a standard choice for each of the ABCD series[31]. If we consider these and work over formal power series  $\mathbb{C}[[\hbar]]$  as in [11], then the pairing between  $\mathbb{A}_q$  and  $\bar{\mathbb{A}}_q^*$  is non-degenerate since this is so ‘near’  $q = 1$ , where our algebras have a classical meaning (in this case the pairing via usual differentiation). Hence we can expect that  $\text{exp}_B$  exists as a formal power series. In this deformation-theoretic setting we can also write  $c = \lambda^\xi$  and the weak quasitriangular structure in Lemma 5.4 becomes  $\mathcal{R}_\xi = \lambda^{-\xi} \otimes \xi$ . Then  $U$  is

quasitriangular with

$$\mathcal{R}_U = \text{exp}_B \lambda^{-\xi \otimes \xi} \mathcal{R}_{U_q(g)}. \quad (47)$$

Moreover, from the relations in Corollary 5.5 we see that  $U$  will also be a deformation of a semisimple Lie algebra. This is because in the limit  $q \rightarrow 1$  (in the sense of [11]) we obtain  $f_i$  and  $e^i$  in the image of  $[\xi, \ ]$  and  $\xi$  from the image of  $[e^i, f_j]$ . Hence we see that applied to standard quantum groups with the standard ‘bosonic’ choice of corresponding quantum plane, the construction in Corollary 5.5 provides a way to construct quantum deformations of  $U(g)$  by induction: the induction step increases the rank of  $g$  by 1 and increases the dimension by  $2n + 1$ , adjoining  $\mathbb{A}_q$  to the positive roots and  $\bar{\mathbb{A}}_q^\star$  to the negative roots. Here  $n$  is the dimension of the defining representation  $g \subseteq M_n$ . From this, we expect that the induction step takes a  $q$ -deformation of  $U(sl_n)$  to one of  $U(sl_{n+1})$ , a  $q$ -deformation of  $U(so_n)$  to  $U(so_{n+1})$  and a  $q$ -deformation of  $U(sp_n)$  to  $U(sp_{n+1})$ .

The same principle applies at our algebraic level. We demonstrate this now on a concrete example, using  $U_q(sl_n)$  in Lusztig’s form in Section 4. For technical reasons (order to use the known weak quasitriangular structure on  $U_q(sl_2)$  in the Drinfeld-Jimbo form) we adjoin the square roots  $K^{\frac{1}{2}}$  to both input and output.

**Example 5.6** *Let  $\mathbb{A}_q$  be the standard bosonic quantum plane of  $sl_2$  type and suppose that  $q$  has a square root in  $k$ . Let  $\dot{U}_q(sl_2)$  denote  $U_q(sl_2)$  from Proposition 4.3 with  $K^{\frac{1}{2}}$  adjoined. This forms a weakly quasitriangular dual pair with  $SL_q(2)$  in a standard form. Then  $U(\mathbb{A}_q, \widetilde{\dot{U}_q(sl_2)}, \bar{\mathbb{A}}_q^\star)$  is  $U_q(sl_3)$  from Proposition 4.3 with  $K^{\frac{1}{2}}$  adjoined.*

**Proof** We start with  $R$ -matrix datum

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix},$$

where the entry at row  $(ik)$  and column  $(jl)$  (taken in the order 11, 12, 21, 22) is  $R^i_j{}^k_l$ . We take  $R' = q^{-2}R$  in (15). This gives the quantum-braided plane  $\mathbb{A}_q^2$  with the usual relations  $e^2e^1 = qe^1e^2$  and the correct braiding  $\Psi$ [14]. The dual  $\bar{\mathbb{A}}_q^\star$  from (15) is similar, with  $f_2f_1 = qf_1f_2$ . The quantum group normalisation for  $R$  needed for the (weak) quasitriangular structure on  $\dot{U}_q(sl_2)$  in [23] is given by  $\lambda = q^{\frac{-3}{2}}$ . The form of  $\mathbf{I}^\pm$  in [31] provides the weak quasitriangular

structure explicitly as

$$\mathbf{m}^+ = \begin{pmatrix} K^{\frac{1}{2}} & -q^{-\frac{1}{2}}(q - q^{-1})eK^{-\frac{1}{2}} \\ 0 & K^{-\frac{1}{2}} \end{pmatrix}, \quad \mathbf{m}^- = \begin{pmatrix} K^{-\frac{1}{2}} & 0 \\ -q^{\frac{1}{2}}(q - q^{-1})K^{\frac{1}{2}}f & K^{\frac{1}{2}} \end{pmatrix}$$

in our present conventions, where  $e, f, K$  are the generators of  $U_q(sl_2)$  in Proposition 4.3.

We first compute the ‘Borel’ relations between  $\mathbf{m}^\pm$  (in the role of ‘Cartan’) and the  $e^i$ . Only five of the entries of  $R$  are non-zero; in particular the only off-diagonal entry is  $R^1_2{}^2_1$ . So we have

$$\begin{aligned} e^1 m^{+1}_1 &= \lambda R^1_1{}^1_1 m^{+1}_1 e^1, & e^1 m^{+1}_2 &= \lambda R^1_1{}^1_2 m^{+1}_2 e^1, & e^2 m^{+2}_2 &= \lambda R^2_2{}^2_2 m^{+2}_2 e^2, \\ e^2 m^{+1}_2 &= \lambda R^1_1{}^2_2 m^{+1}_2 e^2 + \lambda R^1_2{}^2_1 m^{+2}_2 e^1, \\ m^{-2}_1 e^2 &= \lambda R^2_2{}^2_2 e^2 m^{-2}_1, & m^{-2}_1 e^1 &= \lambda R^1_1{}^2_2 e^1 m^{-2}_1 + \lambda R^1_2{}^2_1 e^2 m^{-1}_1, \end{aligned}$$

with the other relations empty or redundant. From the form of  $\mathbf{m}^\pm$  we obtain

$$\begin{aligned} e^1 K^{\frac{1}{2}} &= q^{\frac{1}{2}} K^{\frac{1}{2}} e^1, & e^1 e &= q e e^1, & e^2 K^{-\frac{1}{2}} &= q^{\frac{1}{2}} K^{-\frac{1}{2}} e^2, & q e e^2 - e^2 e &= q^{-\frac{1}{2}} e^1 \\ [f, e^2] &= 0, & [f, e^1] &= -q^{-\frac{1}{2}} K^{-1} e^2 \end{aligned}$$

correspondingly. The calculation for the  $\mathbf{m}^\pm$  relations with  $f_1, f_2$  is similar and the results analogous. The remaining relations from Corollary 5.5 are clearly

$$\begin{aligned} e^i c &= q^{\frac{-3}{2}} c e^i, & c f_i &= q^{\frac{-3}{2}} f_i c, & [e^1, f_1] &= \frac{K^{\frac{1}{2}} c^{-1} - c K^{-\frac{1}{2}}}{q - q^{-1}}, & [e^2, f_2] &= \frac{K^{-\frac{1}{2}} c^{-1} - c K^{\frac{1}{2}}}{q - q^{-1}} \\ [e^1, f_2] &= -q^{-\frac{1}{2}} e K^{-\frac{1}{2}} c^{-1}, & [e^2, f_1] &= q^{\frac{1}{2}} c K^{\frac{1}{2}} f \\ \Delta e^1 &= e^1 \otimes K^{\frac{1}{2}} c^{-1} - q^{-\frac{1}{2}}(q - q^{-1}) e^2 \otimes e K^{-\frac{1}{2}} c^{-1} + 1 \otimes e^1, & \Delta e^2 &= e^2 \otimes K^{-\frac{1}{2}} c^{-1} + 1 \otimes e^2 \\ \Delta f_1 &= f_1 \otimes 1 + c K^{-\frac{1}{2}} \otimes f_1 - q^{\frac{1}{2}}(q - q^{-1})^{-1} c K^{\frac{1}{2}} f \otimes f_2, & \Delta f_2 &= f_2 \otimes 1 + c K^{\frac{1}{2}} \otimes f_2. \end{aligned}$$

Comparing with  $U_q(sl_3)$  in Proposition 4.3 with Cartan matrix  $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$  we see that we can identify  $e, f, K$  as the copy of  $U_q(sl_2)$  associated to  $i = 1$  there, and we can identify  $e^2, f_2, K_2 = K^{-\frac{1}{2}} c^{-1}$  as the copy associated to  $i = 2$  there. We identify the above elements  $e^1, f_1$  from the quantum-braided plane as non-simple roots generated by  $q$ -commutators with  $e, f$ . We construct here  $U_q(sl_3)$  with  $K^{\frac{1}{2}}$  adjoined.

We recover, in fact, more than just  $U_q(sl_3)$  defined by generators and relations (such as the  $q$ -Serre relations contained in Lusztig’s algebra  $f$  in Section 4) – we are explicitly adjoining the non-simple root generators as well. Their relations with the other root generators (which is the content of the  $q$ -Serre relations) appear in our inductive approach from the quantum

plane relations and the ‘Cartan’ relations with  $\mathbf{m}^\pm$ . These are all provided by the inductive construction, along with their explicit coproducts and cross relations.

It is possible to define a sub-Hopf algebra of  $\widetilde{U_q(sl_2)}$  generated by  $\mathbf{m}^\pm c^{\mp 1}$ , and a sub-Hopf algebra of  $\widetilde{SL_q(2)}$  generated by  $\mathbf{t}g$ ; our braided groups live more precisely in the braided category generated by this weakly quasitriangular dual pair. In this way, one can obtain precisely  $U_q(sl_3)$  rather than its extension by  $K^{\frac{1}{2}}$ .  $\square$

We see that our inductive procedure takes us from the standard quantisation to the standard quantisation, at least in this example. In fact, it is clear from another approach[33] that  $U_q(sl_n)$  indeed becomes  $U_q(sl_{n+1})$  when we adjoin its fundamental bosonic quantum plane. This approach[33] provided an inductive construction of the  $R$ -matrix datum and associated quantum matrices by ‘gluing’ quantum planes, but was limited to the  $A$  series or similar (Hecke type)  $R$ -matrices. Our present approach is much more powerful and not limited in this way. From (46) it is also clear that  $U_q(so_n)$  becomes  $U_q(so_{n+2})$ , i.e. the rotation group is turned into the conformal group in the same dimension. The same can be expected for the  $C$  series in its standard  $q$ -deformation.

As a consequence of this inductive approach, we see that the quasitriangular structure of  $U_q(g)$  is then built up by iteration of (47) as a product of braided exponentials in the roots on the left and ‘Gaussian’ factors of the form  $\lambda^{-\xi \otimes \xi}$  which can be collected to the right as the Cartan part. For example, in the deformation-theoretic setting of Example 5.6 we obtain a quasitriangular structure

$$\mathcal{R}_{sl_3} = (\exp_B)_{21}^{-1} \lambda^{-\xi \otimes \xi} \mathcal{R}_{sl_2}; \quad \exp_B = \sum_{m=0}^{\infty} e^{i_1} \cdots e^{i_m} \otimes f_{i_1} \cdots f_{i_m} ([m; q^2]!)^{-1}, \quad (48)$$

where the expression for  $\exp_B$  follows immediately from the free form (14) for any Hecke type  $R$ -matrix[26]. Moreover, if each of the quantum planes being adjoined has a natural basis, we build up by iteration a natural ‘geometrical basis’ of  $U_q(g)$  using the triangular decomposition in Section 3 at each step. Full details of this inductive construction of  $U_q(g)$  at least for the ABCD series will be developed in a sequel.

## A Appendix: Relation with the quantum double of a bosonisation

Drinfeld in [11] introduced a construction for a quasitriangular Hopf algebra  $D(H)$  associated to a (say, finite-dimensional) Hopf algebra  $H$ . Here we use a generalised version  $D(H, A)$  of this[22] associated to a dual pair of Hopf algebras, i.e. two Hopf algebras equipped with a (not necessarily non-degenerate) pairing. They can even be bialgebras so long as the pairing is convolution-invertible[22]. Here  $D(H, A)$  is built on  $H \otimes A$  with the tensor product coalgebra and unit, and the product

$$(h \otimes a)(g \otimes b) = g_{(2)}h \otimes a_{(2)}b \langle Sa_{(1)}, g_{(1)} \rangle \langle a_{(3)}, g_{(3)} \rangle. \quad (49)$$

If  $H$  is a (weakly) quasitriangular Hopf algebra with dual  $A$  and  $B \in \mathcal{M}^A, C \in {}_H\mathcal{M}$  are dually paired braided groups in the sense of a morphism  $\text{ev} : C \otimes B \rightarrow k$  as explained in the Preliminaries, we have a dual bosonisation  $A \bowtie B$  and bosonisation  $C \rtimes H$ , which are dually paired Hopf algebras[25][8] as in (8). We can therefore form their generalised double.

**Lemma A.1** *Let  $B \in \mathcal{M}^A$  and  $C \in {}_H\mathcal{M}$  be categorically dually paired braided groups, where  $H$  is quasitriangular and dually paired with  $A$ . The generalised quantum double  $D(C \rtimes H, A \bowtie B)$  is a Hopf algebra structure on  $C \otimes H \otimes A \otimes B$  with structure*

$$\begin{aligned} (c \otimes h \otimes a \otimes b) \cdot (c' \otimes h' \otimes a' \otimes b') &= (\mathcal{R}_1^{(1)} \triangleright c'_{(2)}) (\mathcal{R}_3^{(2)} h'_{(2)} \triangleright c) \otimes \mathcal{R}_4^{(2)} h'_{(3)} h \otimes a_{(2)} (\mathcal{R}_6^{(2)} h'_{(5)} \triangleright b_{(1)})^{(\bar{2})} a'_{(1)} \\ &\quad \otimes (\mathcal{R}_8^{(1)} \mathcal{R}_7^{(2)} h'_{(6)} \triangleright b_{(2)}) b' \langle \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} h'_{(1)}, Sa_{(1)} \rangle \langle \mathcal{R}_8^{(2)}, a'_{(2)} \rangle \langle \mathcal{R}_5^{(2)} h'_{(4)}, a_{(3)} \rangle \\ &\quad \text{ev}(c'_{(1)}, (\mathcal{R}_6^{(2)} h'_{(5)} \triangleright b_{(1)})^{(\bar{1})}) \text{ev}(\mathcal{R}_7^{(1)} \mathcal{R}_6^{(1)} \mathcal{R}_5^{(1)} \mathcal{R}_4^{(1)} \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \triangleright \underline{S}^{-1} c'_{(3)}, h'_{(7)} \triangleright b_{(3)}), \\ \Delta(c \otimes h \otimes a \otimes b) &= c_{(1)} \otimes \mathcal{R}^{(2)} h_{(1)} \otimes a_{(1)} \otimes b_{(1)}^{(\bar{1})} \otimes \mathcal{R}^{(1)} \triangleright c_{(2)} \otimes h_{(2)} \otimes a_{(2)} b_{(1)}^{(\bar{2})} \otimes b_{(2)}. \end{aligned}$$

Similarly when  $H, A$  are a weakly quasitriangular dual pair. In the finite-dimensional non-degenerately paired case we have Drinfeld's quasitriangular structure as

$$\mathcal{R}_D = (f^\alpha u \mathcal{R}^{(1)} \triangleright \underline{S} f^\alpha \otimes f^\beta S \mathcal{R}^{(2)} \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes e_\alpha \otimes e_\alpha e_\beta)$$

where  $\{e_\alpha\}$  is a basis of  $B$  with dual  $\{f^\alpha\}$  and  $\{e_\alpha\}$  a basis of  $A$  with dual  $\{f^\alpha\}$ .

**Proof** This is a straightforward but tedious calculation. We emphasise the case where  $H$  is quasitriangular because the notation is more familiar; as explained in Remark 3.9, our proofs and results convert immediately over to the weakly quasitriangular case. The structure of  $C \bowtie H$  is in (6) and its dual construction  $A \bowtie B$  is in (8). We need their iterated coproducts as

$$\begin{aligned} (\text{id} \otimes \Delta) \circ \Delta(c \otimes h) &= \underline{c}_{(1)} \otimes \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} h_{(1)} \otimes \mathcal{R}_1^{(1)} \triangleright \underline{c}_{(2)} \otimes \mathcal{R}_3^{(2)} h_{(2)} \otimes \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \triangleright \underline{c}_{(3)} \otimes h_{(3)} \\ (\Delta \otimes \text{id}) \circ \Delta(a \otimes b) &= a_{(1)} \otimes \underline{b}_{(1)}^{(\bar{1})} \otimes a_{(2)} \underline{b}_{(1)}^{(\bar{2})}{}_{(1)} \otimes \underline{b}_{(2)}^{(\bar{1})} \otimes a_{(3)} \underline{b}_{(1)}^{(\bar{2})}{}_{(2)} \underline{b}_{(2)}^{(\bar{2})} \otimes \underline{b}_{(3)}. \end{aligned}$$

Then

$$\begin{aligned} (c \otimes h \otimes a \otimes b) \cdot (c' \otimes h' \otimes a' \otimes b') &= \\ &= (c' \otimes h')_{(2)} \cdot (c \otimes h) \otimes (a \otimes b)_{(2)} \cdot (a' \otimes b') \langle S(a \otimes b)_{(1)}, (c' \otimes h')_{(1)} \rangle \langle (a \otimes b)_{(3)}, (c' \otimes h')_{(3)} \rangle \\ &= (\mathcal{R}_1^{(1)} \triangleright \underline{c}'_{(2)} \otimes \mathcal{R}_3^{(2)} h'_{(2)}) \cdot (c \otimes h) \otimes (a_{(2)} \otimes \underline{b}_{(1)}^{(\bar{2})}{}_{(1)} \otimes \underline{b}_{(2)}^{(\bar{1})}) \cdot (a' \otimes b') \\ &\quad \langle S(\mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} h'_{(1)}), a_{(1)} \rangle \langle \mathcal{R}_4^{(2)} h'_{(3)}, a_{(3)} \underline{b}_{(1)}^{(\bar{2})}{}_{(2)} \underline{b}_{(2)}^{(\bar{2})} \rangle \\ &\quad \text{ev}(\underline{c}'_{(1)}, \underline{b}_{(1)}^{(\bar{1})}) \text{ev}(\mathcal{R}_4^{(1)} \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \triangleright \underline{S}^{-1} \underline{c}'_{(3)}, h'_{(4)} \triangleright \underline{b}_{(3)}) \\ &= (\mathcal{R}_1^{(1)} \triangleright \underline{c}'_{(2)}) (\mathcal{R}_3^{(2)} h'_{(2)} \triangleright c) \otimes \mathcal{R}_4^{(2)} h'_{(3)} h \otimes a_{(2)} \underline{b}_{(1)}^{(\bar{2})}{}_{(1)} a'_{(1)} \otimes \underline{b}_{(2)}^{(\bar{1})} b' \\ &\quad \langle \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} h'_{(1)}, S a_{(1)} \rangle \langle \mathcal{R}_5^{(2)} h'_{(4)}, a_{(3)} \underline{b}_{(1)}^{(\bar{2})}{}_{(2)} \underline{b}_{(2)}^{(\bar{2})}{}_{(2)} \rangle \langle \mathcal{R}_6^{(1)}, \underline{b}_{(2)}^{(\bar{2})}{}_{(1)} \rangle \langle \mathcal{R}_6^{(2)}, a'_{(2)} \rangle \\ &\quad \text{ev}(\underline{c}'_{(1)}, \underline{b}_{(1)}^{(\bar{1})}) \text{ev}(\mathcal{R}_5^{(1)} \mathcal{R}_4^{(1)} \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \triangleright \underline{S}^{-1} \underline{c}'_{(3)}, h'_{(5)} \triangleright \underline{b}_{(3)}) \end{aligned}$$

using the definition (49) applied to our case, the triple coproducts above and then the products from (6) and (8). We also used invariance of  $\text{ev}$  in the form  $\text{ev}(c, h \triangleright b) = \text{ev}(S^{-1} h \triangleright c, b)$  and the properties of actions and coactions. Further rewriting the right coaction as a left action  $b^{(\bar{1})} \langle h, b^{(\bar{2})} \rangle = h \triangleright b$ , gives the formula stated. The coproduct of the quantum double is the tensor product one from (6)–(8).

For the quasitriangular structure in the case  $A = H^*, C = B^*$ , we see from the duality pairing (8) that if  $e_{\alpha, a} = e_{\alpha} \otimes e_b$  is a basis of  $A \otimes B$  then

$$\begin{aligned} f^{\alpha, a} &= S(f^a \otimes S^{-1} f^{\alpha}) = (S(S^{-1} f^{\alpha})_{(2)}) u \mathcal{R}^{(1)} \triangleright \underline{S} f^a \otimes S(\mathcal{R}^{(2)}(S^{-1} f^{\alpha})_{(1)}) \\ &= f^{\alpha}{}_{(1)} u \mathcal{R}^{(1)} \triangleright \underline{S} f^a \otimes f^{\alpha}{}_{(2)} S \mathcal{R}^{(2)} \end{aligned}$$

is a dual basis under the pairing. We computed the antipode (6) of  $C \bowtie H$  from [7]. Then Drinfeld's quasitriangular structure  $\mathcal{R} = f^{\alpha, a} \otimes 1 \otimes 1 \otimes e_{\alpha, a}$  giving the formula shown. We have emphasised this case. When  $A, H$  and  $C, B \in \mathcal{M}^A$  are merely dually paired then the quantum

double here is weakly quasitriangular with respect to the natural codouble construction made along the same lines as above.  $\square$

We now apply this general construction to the specific setting of Section 3. There we had a braided group  $B \in {}^A\mathcal{M}$  which we consider as  $B \in \mathcal{M}^{A^{\text{cop}}}$ , and categorically dually paired with it a braided group  $C$  in  $\mathcal{M}_H$  which we consider as  $C \in {}_{H^{\text{op}}}\mathcal{M}$ .

**Theorem A.2** *In Section 3, there is a Hopf algebra surjection  $\pi : D(C \bowtie H^{\text{op}}, A^{\text{cop}} \ltimes B) \rightarrow U(\bar{C}, H, B)$  from the generalised quantum double, given by*

$$\pi(c \otimes h \otimes a \otimes b) = c \triangleleft S h_{(1)} \otimes h_{(2)} \mathcal{R}^{(2)} \langle a, \mathcal{R}^{(1)} \rangle \otimes b.$$

*In the finite-dimensional non-degenerately paired case this is a surjection of quasitriangular Hopf algebras.*

**Proof** First, we write out the product from Lemma A.1 applied to our particular dual pair. We equip  $H^{\text{op}}$  with antipode  $S^{\text{op}} = S^{-1}$  and quasitriangular structure  $\mathcal{R}^{\text{op}} = \mathcal{R}_{21}$ . Then we write everything in terms of the usual structure of  $H$  and the usual coproduct of  $A$ . We write the left action of  $H^{\text{op}}$  on  $C$  in its original form as a right action of  $H$  (as in Section 3), and the right coaction of  $A^{\text{cop}}$  in its original form as a left coaction of  $A$  or (by evaluation) a right action of  $H$ . These are simple manipulations on the formulae in Lemma A.1. The resulting product between various combinations of the tensor factors (embedded in  $C \otimes H \otimes A \otimes B$  in the trivial way by tensoring with 1) becomes

$$\begin{aligned} (h \otimes a)(h' \otimes a') &= hh'_{(2)} \otimes a_{(2)} a' \langle h'_{(3)}, a_{(1)} \rangle \langle h'_{(1)}, S^{-1} a_{(3)} \rangle \\ (c \otimes h)(c' \otimes h') &= c'(c \triangleleft h'_{(1)}) \otimes hh'_{(2)}, \quad (a \otimes b)(a' \otimes b') = aa'_{(2)} \otimes (b \triangleleft \mathcal{R}^{(2)}) b' \langle \mathcal{R}^{(1)}, a'_{(1)} \rangle \\ (c \otimes a)(c' \otimes a') &= (c' \triangleleft \mathcal{R}^{(2)}) c \otimes a_{(1)} a' \langle \mathcal{R}^{(1)}, S^{-1} a_{(2)} \rangle, \quad (h \otimes b)(h' \otimes b') = hh'_{(1)} \otimes (b \triangleleft h'_{(2)}) b' \\ (c \otimes b)(c' \otimes b') &= c'_{(2)} (c \triangleleft \mathcal{R}_1^{(1)}) \otimes \mathcal{R}_2^{(1)} \otimes (b_{(1)} \triangleleft \mathcal{R}_3^{(1)})^{(\bar{1})} \otimes (b_{(2)} \triangleleft \mathcal{R}_4^{(1)}) b' \\ &\quad \text{ev}(c'_{(1)}, (b_{(1)} \triangleleft \mathcal{R}_3^{(1)})^{(\bar{2})}) \text{ev}(S^{-1} c'_{(3)} \triangleleft \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_3^{(2)} \mathcal{R}_4^{(2)}, b_{(3)}), \end{aligned} \tag{50}$$

while the coproduct becomes

$$\Delta(c \otimes h \otimes a \otimes b) = c_{(1)} \otimes h_{(1)} \mathcal{R}^{(1)} \otimes a_{(2)} \otimes b_{(1)}^{(\bar{2})} \otimes c_{(2)} \triangleleft \mathcal{R}^{(2)} \otimes h_{(2)} \otimes a_{(1)} b_{(1)}^{(\bar{1})} \otimes b_{(2)}. \tag{51}$$

We recognise a version of the generalised quantum double built on  $H \otimes A^{\text{cop}}$  as a sub-Hopf algebra. We also recognise sub-Hopf algebras  $(C \bowtie H^{\text{op}})^{\text{op}}$  and  $A^{\text{cop}} \ltimes B$  as expected. Finally,



we recognise that  $H \bowtie B \subseteq U$  appears here as a subalgebra, as does  $\bar{C} \rtimes \bar{H}$  up to an elementary isomorphism.

We verify now that  $\pi$  is a Hopf algebra map to  $U$  from Section 3 (it is clearly surjective). The restriction to the generalised quantum double of  $H, A$  is  $\pi(h \otimes a) = h\mathcal{R}^{(1)}\langle \mathcal{R}^{(2)}, a \rangle$  which is a version of the Hopf algebra projection  $D(H) \rightarrow H$  introduced in [24] when  $H$  is quasitriangular. The restriction to  $C \otimes H$  is  $\pi(c \otimes h) = c \triangleleft S h_{(1)} \otimes h_{(2)} = h_{(1)} \triangleright c \otimes h_{(2)}$  and gives an isomorphism to the algebra  $\bar{C} \rtimes \bar{H} \subseteq U$ . We use elementary properties of Hopf algebras and the conversion of the action on  $\bar{C}$  as in (26). The restriction of  $\pi$  to  $H \otimes B$  is the identity and hence immediately an algebra map to  $U$ . For the remaining restrictions, we have

$$\begin{aligned} \pi((a \otimes b)(a' \otimes b')) &= \langle aa'_{(2)}, \mathcal{R}_1^{(1)} \rangle \mathcal{R}_1^{(2)} \otimes (b \triangleleft \mathcal{R}_3^{(2)}) b' \langle \mathcal{R}_3^{(1)}, a'_{(1)} \rangle \\ &= \langle a, \mathcal{R}_1^{(1)} \rangle \langle a', \mathcal{R}_3^{(1)} \mathcal{R}_2^{(1)} \rangle \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} \otimes (b \triangleleft \mathcal{R}_3^{(2)}) b' \\ &= \langle a, \mathcal{R}_1^{(1)} \rangle \langle a', \mathcal{R}_2^{(1)} \rangle (\mathcal{R}_1^{(2)} \otimes b) \cdot (\mathcal{R}_2^{(2)} \otimes b') = (\pi(a \otimes b))(\pi(a' \otimes b')), \end{aligned}$$

using the elementary properties of the quasitriangular structure and the relations of  $H \bowtie B \subseteq U$ .

Likewise, we have

$$\begin{aligned} \pi((c \otimes a)(c' \otimes a')) &= (c' \triangleleft \mathcal{R}_1^{(2)}) \cdot_C c \otimes \mathcal{R}_2^{(2)} \langle a_{(1)} a', \mathcal{R}_2^{(1)} \rangle \langle \mathcal{R}_1^{(1)}, S^{-1} a_{(2)} \rangle \\ &= c(\mathcal{R}_1^{(2)} \triangleright c') \otimes \mathcal{R}_2^{(2)} \mathcal{R}_3^{(2)} \langle a, \mathcal{R}_2^{(1)} \mathcal{R}_1^{(1)} \rangle \langle a', \mathcal{R}_3^{(1)} \rangle \\ &= (c \otimes \mathcal{R}_1^{(2)}) \cdot (c' \otimes \mathcal{R}_3^{(2)}) \langle \mathcal{R}_1^{(1)}, a \rangle \langle \mathcal{R}_3^{(1)}, a' \rangle = (\pi(c \otimes a))(\pi(c' \otimes a')), \end{aligned}$$

where the latter products are in  $\bar{C}$  (which has the opposite algebra structure to  $C$ ) and  $H$ , or, finally, in  $\bar{C} \rtimes \bar{H} \subseteq U$ . We used invariance of  $\mathcal{R}$  under  $S \otimes S$ . For the final restriction, we need the iteration of the relation (26) between the coproducts of  $C, \bar{C}$  as

$$(\bar{\Delta} \otimes \text{id}) \circ \bar{\Delta} c = c_{(1)} \triangleleft \mathcal{R}_3^{- (1)} \mathcal{R}_8^{- (1)} \otimes c_{(2)} \triangleleft \mathcal{R}_7^{- (1)} \mathcal{R}_8^{- (2)} \otimes c_{(3)} \triangleleft \mathcal{R}_7^{- (2)} \mathcal{R}_3^{- (2)}, \quad (52)$$

where the numbering of the copies of  $\mathcal{R}^{-1}$  is to keep them distinct from other copies used in the proof. Then

$$\begin{aligned} \pi((c \otimes b)(c' \otimes b')) &= (c'_{(2)} \cdot_C (c \triangleleft \mathcal{R}_1^{(1)})) \triangleleft S \mathcal{R}_2^{(1)} \otimes \mathcal{R}_5^{(1)} \mathcal{R}_6^{(2)} \langle \mathcal{R}_6^{(1)}, (b_{(1)} \triangleleft \mathcal{R}_3^{(1)})^{\bar{(1)}} \rangle \otimes (b_{(2)} \triangleleft \mathcal{R}_4^{(1)}) b' \\ &\quad \text{ev}(c'_{(1)}, (b_{(1)} \triangleleft \mathcal{R}_3^{(1)})^{\bar{(2)}}) \text{ev}(\underline{S}^{-1} c'_{(3)} \triangleleft \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} \mathcal{R}_5^{(2)} \mathcal{R}_3^{(2)} \mathcal{R}_4^{(2)}, b_{(3)}) \\ &= (c'_{(2)} \triangleleft \mathcal{R}_7^{- (1)}) \cdot_C c \otimes \mathcal{R}_6^{(1)} \mathcal{R}_5^{(1)} \otimes (b_{(2)} \triangleleft \mathcal{R}_4^{(1)}) b' \end{aligned}$$

$$\begin{aligned}
& \text{ev}(c'_{\underline{(1)}}, b_{\underline{(1)}} \triangleleft \mathcal{R}_6^{(1)} \mathcal{R}_3^{(1)}) \text{ev}(\underline{S}^{-1} c_{\underline{(3)}} \triangleleft \mathcal{R}_7^{-(2)} \mathcal{R}_3^{(2)} \mathcal{R}_5^{(2)} \mathcal{R}_4^{(2)}, b_{\underline{(3)}}) \\
&= c(c'_{\underline{(2)}} \triangleleft \mathcal{R}_8^{(2)}) \otimes \mathcal{R}_6^{(2)} \mathcal{R}_5^{(1)} \otimes (b_{\underline{(2)}} \triangleleft \mathcal{R}_4^{(1)}) b' \\
& \text{ev}(c'_{\underline{(\bar{1})}} \triangleleft \mathcal{R}_8^{(1)}, b_{\underline{(1)}} \triangleleft \mathcal{R}_6^{(1)}) \text{ev}(\bar{S} c'_{\underline{(\bar{3})}} \triangleleft \mathcal{R}_5^{(2)} \mathcal{R}_4^{(2)}, b_{\underline{(3)}}) \\
&= (\pi(c \otimes b))(\pi(c' \otimes b')).
\end{aligned}$$

For the second equality we evaluated the coaction on  $b_{\underline{(1)}} \triangleleft \mathcal{R}_3^{(1)}$  as an action on it of  $\mathcal{R}_6^{(1)}$ , cancelled  $\mathcal{R}_1 \mathcal{R}_2^{-1}$  and then used the QYBE applied to  $\mathcal{R}_3, \mathcal{R}_6, \mathcal{R}_5$ . The third equality is invariance of  $\text{ev}$  and (52). We also adopt the product of  $\bar{C}$ . Making the further notational change from  $\text{ev}$  to  $\langle \cdot, \cdot \rangle$  as in Section 3, using invariance under  $\mathcal{R}_4^{(2)}$  and writing the action of  $\mathcal{R}_8^{(1)}$  from the right as  $S\mathcal{R}_8^{(1)}$  from the left by (26) allows us to recognise the product in  $U$  as the final step.

To see that  $\pi$  is a coalgebra map, we compute

$$\begin{aligned}
(\pi \otimes \pi)\Delta(c \otimes h \otimes a \otimes b) &= c_{\underline{(1)}} \triangleleft S\mathcal{R}_1^{(1)} Sh_{(1)} \otimes h_{(2)} \mathcal{R}_2^{(1)} h_{(2)} \mathcal{R}_3^{(2)} \otimes b_{\underline{(1)}}^{(\bar{2})} \langle a_{(2)}, \mathcal{R}_3^{(1)} \rangle \\
& \otimes c_{\underline{(2)}} \triangleleft \mathcal{R}_1^{(2)} \mathcal{R}_2^{(2)} Sh_{(3)} \otimes h_{(4)} \mathcal{R}_4^{(2)} \otimes b_{\underline{(2)}} \langle a_{(1)} b_{(1)}^{(\bar{1})}, \mathcal{R}_4^{(1)} \rangle \\
&= h_{(1)} \triangleright c_{\underline{(\bar{1})}} \otimes \mathcal{R}_2^{(1)} h_{(3)} \mathcal{R}_3^{(2)} \otimes b_{\underline{(1)}} \triangleleft \mathcal{R}_5^{(1)} \langle a, \mathcal{R}_4^{(1)} \mathcal{R}_3^{(1)} \rangle \otimes h_{(2)} S^{-1} \mathcal{R}_2^{(2)} \triangleright c_{\underline{(2)}} \otimes h_{(4)} \mathcal{R}_4^{(2)} \mathcal{R}_5^{(2)} \otimes b_{\underline{(2)}} \\
&= \Delta_U(h_{(1)} \triangleright c \otimes h_{(2)} \mathcal{R}^{(2)} \langle a, \mathcal{R}^{(1)} \rangle \otimes b) = \Delta_U \circ \pi(c \otimes h \otimes a \otimes b)
\end{aligned}$$

as required. The first equality is the definitions and elementary properties (1). The second equality is (26). We also write the action of  $Sh_{(1)}$  on  $C$  from the right as  $h_{(1)}$  from the left, etc. In addition, we use the quasicoassociativity axiom (1) to  $h_{(2)} \otimes h_{(3)}$ . We then identify the result in terms of the coproduct of  $U$ .

Finally, in the finite-dimensional non-degenerately paired case we have the quasitriangular structure converted from Lemma A.1 (as explained above) as

$$\mathcal{R}_D = (\underline{S} f^a \triangleleft (S\mathcal{R}^{(2)}) u f^\alpha \otimes \mathcal{R}^{(1)} f^\beta \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes e_\alpha e_\beta \otimes e_a).$$

We used invariance of  $\mathcal{R}$  under  $S \otimes S$ . Then

$$\begin{aligned}
(\pi \otimes \pi)(\mathcal{R}_D) &= \underline{S} f^a \triangleleft (S\mathcal{R}_1^{(2)}) \mathcal{R}_2^{(2)} u f^\alpha (S f^\beta) \mathcal{R}_2^{(1)} \otimes \mathcal{R}_1^{(1)} f^\gamma \otimes 1 \otimes 1 \otimes \mathcal{R}_3^{(2)} \langle e_\alpha e_\beta e_\gamma, \mathcal{R}_3^{(1)} \rangle \otimes e_a \\
&= \mathcal{R}_1^{(2)} \triangleright \bar{S}^{-1} f^a \otimes \mathcal{R}_1^{(1)} \mathcal{R}_3^{(1)} \otimes 1 \otimes 1 \otimes \mathcal{R}_3^{(2)} \otimes e_a = \mathcal{R}_U
\end{aligned}$$

using the definition of  $\pi$ , the antipode axioms followed by  $\mathcal{R}^{(2)} u \mathcal{R}^{(1)} = 1$ . We write the action of  $S\mathcal{R}_1^{(2)}$  from the right as an action from the left. Using invariance of the coevaluation  $f^a \otimes e_a$  and changing to a new basis  $f'^a = \bar{S}^{-1} f^a$  with dual  $\bar{S} e'_a$  gives  $\mathcal{R}_U$  from Proposition 3.6.  $\square$

## B Appendix: Double biproducts

This appendix introduces a ‘double biproduct’ construction which generalises the double bosonisation in Section 3. We recall that Radford in [16] considered as ‘biproducts’ the general class of Hopf algebras which are a smash (or cross) product and coproduct by the simultaneous action and coaction of a Hopf algebra  $H$ ; it is clear from [9][24] that bosonisations can be viewed as examples of this general form (see the Preliminaries). We extend this observation now to our ‘doubled’ setting by introducing the required ‘double’ construction. This provides an alternative point of view because the focus is now on actions and coactions (rather than on quasitriangular and dual-quasitriangular structures), which should be more accessible to the algebraically minded reader. The natural setting in the present section is with  $H$  a general Hopf algebra with bijective antipode. The antipode and its inverse are not actually needed in the main construction, i.e. the result is slightly more general.

The correct setting for biproducts was identified in [24][9] as the braided category of crossed modules  $\mathcal{M}_H^H$ , as discussed independently in [34] in another context. This category is in fact nothing other than a version of the braided category of  $D(H)$ -modules introduced through the work of V.G. Drinfeld. Simply, one casts the action of  $H^* \subseteq D(H)$  as a coaction of  $H$ ; this is then well-defined even for  $H$  infinite-dimensional. An object in  $\mathcal{M}_H^H$  is a vector space on which  $H$  acts and coacts from the right such that

$$v^{(\bar{1})} \triangleleft h_{(1)} \otimes v^{(\bar{2})} h_{(2)} = (v \triangleleft h_{(2)})^{(\bar{1})} \otimes h_{(1)} (v \triangleleft h_{(2)})^{(\bar{2})}. \quad (53)$$

A morphism is a linear map intertwining both the action and coaction of  $H$ . There is also a left-handed version  ${}^H_H\mathcal{M}$  with

$$h_{(1)} v^{(\bar{1})} \otimes h_{(2)} \triangleright v^{(\bar{2})} = (h_{(1)} \triangleright v)^{(\bar{1})} h_{(2)} \otimes (h_{(1)} \triangleright v)^{(\bar{2})}. \quad (54)$$

Both categories are braided, with

$$\Psi_{V,W}(v \otimes w) = w^{(\bar{1})} \otimes v \triangleleft w^{(\bar{2})}, \quad \Psi_{V,W}(v \otimes w) = v^{(\bar{1})} \triangleright w \otimes v^{(\bar{2})} \quad (55)$$

for the two cases. These are just the braidings corresponding to Drinfeld’s quasitriangular structure on the quantum double. As explained by the author in [24][9], braided groups in such categories exactly satisfy the conditions in [16] to make a simultaneous cross product and cross

coproduct by the given action and coaction, and obtain an ordinary Hopf algebra. We consider braided groups  $B \in \mathcal{M}_H^H$  and  $\bar{C} \in {}^H_H\mathcal{M}$  and denote the corresponding biproducts by  $H \bowtie B$  and  $\bar{C} \rtimes H$  as an extension of our previous notation. We assume, moreover, that there is a ‘pairing’  $\langle \cdot, \cdot \rangle : \bar{C} \otimes B \rightarrow k$  such that

$$\begin{aligned} \langle h \triangleright c, b \rangle &= \langle c, b \triangleleft h \rangle \\ \langle c, ab \rangle &= \langle c_{(\bar{2})}, a \triangleleft b^{(\bar{2})} \rangle \langle c_{(\bar{1})}, b^{(\bar{1})} \rangle, \quad \langle cd, b \rangle = \langle c, b_{(\underline{1})} \rangle \langle d, b_{(\underline{2})} \rangle, \quad \langle \bar{S}c, b \rangle = \langle c, \underline{S}^{-1}b \rangle \end{aligned} \quad (56)$$

for all  $h \in H$ ,  $a, b \in B$  and  $c, d \in \bar{C}$ . We deduce (from braided antimultiplicativity of braided antipodes[6]) that

$$\langle \bar{S}c, ab \rangle = \langle \bar{S}c_{(\bar{1})}, a \rangle \langle \bar{S}c_{(\bar{2})}, b \rangle, \quad \langle \bar{S}(cd), b \rangle = \langle c^{(\bar{1})} \triangleright \bar{S}d, b_{(\underline{1})} \rangle \langle \bar{S}c^{(\bar{2})}, b_{(\underline{2})} \rangle \quad (57)$$

hold as well. The first condition in (56) expresses bicovariance of the pairing under the action. In place of bicovariance under the coaction, however, we adopt a compatibility condition

$$b^{(\bar{1})} \triangleleft c^{(\bar{1})} \otimes b^{(\bar{2})} \triangleright c^{(\bar{2})} = b \otimes c, \quad \forall b \in B, c \in \bar{C} \quad (58)$$

between the two crossed module structures. This is the data needed for the following construction.

**Theorem B.1** *Let  $H$  be a Hopf algebra with invertible antipode,  $B \in \mathcal{M}_H^H$ ,  $\bar{C} \in {}^H_H\mathcal{M}$  braided groups obeying (58), and  $\langle \cdot, \cdot \rangle$  a braided skew-pairing as above. Then there is a unique ordinary Hopf algebra  $\mathcal{U}(\bar{C}, H, B)$  built on  $\bar{C} \otimes H \otimes B$ , the double biproduct, containing  $H \bowtie B$  and  $\bar{C} \rtimes H$  as sub-Hopf algebras with cross relations*

$$bc = b_{(\underline{1})}^{(\bar{2})} c_{(\bar{2})} b_{(\underline{2})} c_{(\bar{3})}^{(\bar{1})} \langle c_{(\bar{1})}, b_{(\underline{1})}^{(\bar{1})} \rangle \langle \bar{S}c_{(\bar{3})}^{(\bar{2})}, b_{(\underline{3})} \rangle$$

**Proof** We provide an outline, following the same strategy as in the proofs of Theorem 3.2. As before, we work with the cross relations in a more canonical form

$$bc = (b_{(\underline{1})}^{(\bar{2})})_{(1) \triangleright c_{(\bar{2})}} b_{(\underline{1})}^{(\bar{2})} c_{(\bar{3})}^{(\bar{1})} (b_{(\underline{2})} \triangleleft c_{(\bar{3})}^{(\bar{1})})_{(2)} \langle c_{(\bar{1})}, b_{(\underline{1})}^{(\bar{1})} \rangle \langle \bar{S}c_{(\bar{3})}^{(\bar{2})}, b_{(\underline{3})} \rangle, \quad (59)$$

in view of the relations of  $H \bowtie B$  and  $\bar{C} \rtimes H$ . This has the form  $bc = \sum c_i R_i b_i$  for  $c_i \in \bar{C}$ ,  $R_i \in H$  and  $b_i \in B$ . The general product is defined from this as in Section 3, and comes out as

$$(chb) \cdot (dga) = c(h_{(1)} b_{(\underline{1})}^{(\bar{2})})_{(1) \triangleright d_{(\bar{2})}} (h_{(2)} b_{(\underline{1})}^{(\bar{2})} d_{(\bar{3})}^{(\bar{1})})_{(1)} g_{(1)} (b_{(\underline{2})} \triangleleft d_{(\bar{3})}^{(\bar{1})})_{(2)} g_{(2)} a \langle d_{(\bar{1})}, b_{(\underline{1})}^{(\bar{1})} \rangle \langle \bar{S}d_{(\bar{3})}^{(\bar{2})}, b_{(\underline{3})} \rangle. \quad (60)$$

The argument that it is enough to prove associativity of the products  $(a \cdot (chb)) \cdot d = a \cdot ((chb) \cdot d)$  goes through in the same way (it requires only covariance of the algebras of  $B, \bar{C}$  under  $H$ ). To prove this special case we compute both sides from (60). The left hand side is

$$\begin{aligned}
(a \cdot (chb)) \cdot d &= \left( (a_{(1)}^{(\bar{2})}{}_{(1)} \triangleright c_{(\bar{2})}) (a_{(1)}^{(\bar{2})}{}_{(2)} c_{(\bar{3})}^{(\bar{1})}{}_{(1)} h_{(1)}) ((a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(2)} h_{(2)}) b) \right) \cdot d \langle c_{(\bar{1})}, a_{(1)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{3})}^{(\bar{2})}, a_{(3)} \rangle \\
&= (a_{(1)}^{(\bar{2})}{}_{(1)} \triangleright c_{(\bar{2})}) (a_{(1)}^{(\bar{2})}{}_{(2)} c_{(\bar{3})}^{(\bar{1})}{}_{(1)} h_{(1)}) ((a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)}) b)_{(1)}^{(\bar{2})}{}_{(1)} \triangleright d_{(\bar{2})}) \\
&\quad \otimes a_{(1)}^{(\bar{2})}{}_{(3)} c_{(\bar{3})}^{(\bar{1})}{}_{(2)} h_{(2)} ((a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)}) b)_{(1)}^{(\bar{2})}{}_{(2)} d_{(\bar{3})}^{(\bar{1})}{}_{(1)} \otimes ((a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)}) b)_{(2)} \triangleleft d_{(\bar{3})}^{(\bar{1})}{}_{(2)} \\
&\quad \langle c_{(\bar{1})}, a_{(1)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{3})}^{(\bar{2})}, a_{(3)} \rangle \langle d_{(\bar{1})}, ((a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)}) b)_{(1)}^{(\bar{1})} \rangle \langle \bar{S} d_{(\bar{3})}^{(\bar{2})}, ((a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)}) b)_{(\bar{3})} \rangle \\
&= (a_{(1)}^{(\bar{2})}{}_{(1)} \triangleright c_{(\bar{2})}) (a_{(1)}^{(\bar{2})}{}_{(2)} c_{(\bar{3})}^{(\bar{1})}{}_{(1)} h_{(1)} (a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)})^{(\bar{2})}{}_{(1)} b_{(1)}^{(\bar{2})}{}_{(2)} \triangleright d_{(\bar{3})}) \\
&\quad \otimes a_{(1)}^{(\bar{2})}{}_{(3)} c_{(\bar{3})}^{(\bar{1})}{}_{(2)} h_{(2)} (a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)})^{(\bar{2})}{}_{(2)} b_{(1)}^{(\bar{2})}{}_{(3)} d_{(\bar{4})}^{(\bar{1})}{}_{(1)} d_{(\bar{5})}^{(\bar{1})}{}_{(1)} \\
&\quad \otimes (a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(4)} h_{(4)} b_{(1)}^{(\bar{2})}{}_{(4)} d_{(\bar{4})}^{(\bar{1})}{}_{(2)} d_{(\bar{5})}^{(\bar{1})}{}_{(2)}) (b_{(2)} \triangleleft d_{(\bar{4})}^{(\bar{1})}{}_{(3)} d_{(\bar{5})}^{(\bar{1})}{}_{(3)}) \langle c_{(\bar{1})}, a_{(1)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{3})}^{(\bar{2})}, a_{(5)} \rangle \\
&\quad \langle d_{(\bar{2})}, (a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)})^{(\bar{1})} \triangleleft b_{(1)}^{(\bar{2})}{}_{(1)} \rangle \langle d_{(\bar{1})}, b_{(1)}^{(\bar{1})} \rangle \langle \bar{S} d_{(\bar{4})}^{(\bar{2})}, a_{(4)} \triangleleft b_{(1)}^{(\bar{2})}{}_{(5)} b_{(2)}^{(\bar{2})} \rangle \langle \bar{S} d_{(\bar{5})}^{(\bar{2})}, b_{(3)} \rangle,
\end{aligned}$$

where the first two equalities are two applications of the product from (60). The third equality then puts in the iterated braided coproduct of a product in  $B$  from the first half of

$$\begin{aligned}
(\underline{\Delta} \otimes \text{id}) \circ \underline{\Delta}(ab) &= a_{(1)} b_{(1)}^{(\bar{1})} \otimes (a_{(2)} \triangleleft b_{(1)}^{(\bar{2})}{}_{(1)}) b_{(2)}^{(\bar{1})} \otimes (a_{(3)} \triangleleft b_{(1)}^{(\bar{2})}{}_{(2)} b_{(2)}^{(\bar{2})}) b_{(3)} \\
(\text{id} \otimes \bar{\Delta}) \circ \bar{\Delta}(cd) &= c_{(\bar{1})} (c_{(\bar{2})}^{(\bar{1})} c_{(\bar{3})}^{(\bar{1})}{}_{(1)} \triangleright d_{(\bar{1})}) \otimes c_{(\bar{2})}^{(\bar{2})} (c_{(\bar{3})}^{(\bar{1})}{}_{(2)} \triangleright d_{(\bar{2})}) \otimes c_{(\bar{3})}^{(\bar{2})} d_{(\bar{3})}.
\end{aligned} \tag{61}$$

We also use covariance of the products and coproducts under the action and coaction of  $H$ , and break down the pairing with products from  $B$  using (56)–(57). This gives the expression above. We now use bicovariance of the pairing to move  $\triangleleft b_{(2)}^{(\bar{2})}$  to act on  $d_{(\bar{4})}^{(\bar{2})}$  and (58) to cancel it, and we use the crossing condition (53) on  $\Delta(c_{(\bar{3})}^{(\bar{1})}{}_{(1)} h_{(1)} (a_{(2)} \triangleleft c_{(\bar{2})}^{(\bar{1})}{}_{(3)} h_{(2)})^{(\bar{2})}) \otimes (a_{(2)} \triangleleft c_{(\bar{2})}^{(\bar{1})}{}_{(3)} h_{(2)})^{(\bar{1})}$ . These steps give

$$\begin{aligned}
(a \cdot (chb)) \cdot d &= (a_{(1)}^{(\bar{2})}{}_{(1)} \triangleright c_{(\bar{2})}) (a_{(1)}^{(\bar{2})}{}_{(2)} a_{(2)}^{(\bar{2})}{}_{(1)} c_{(\bar{3})}^{(\bar{1})}{}_{(2)} h_{(2)} b_{(1)}^{(\bar{2})}{}_{(2)} \triangleright d_{(\bar{3})}) \\
&\quad \otimes a_{(1)}^{(\bar{2})}{}_{(3)} a_{(2)}^{(\bar{2})}{}_{(2)} c_{(\bar{3})}^{(\bar{1})}{}_{(3)} h_{(3)} b_{(1)}^{(\bar{2})}{}_{(3)} d_{(\bar{4})}^{(\bar{1})}{}_{(1)} d_{(\bar{5})}^{(\bar{1})}{}_{(1)} \\
&\quad \otimes (a_{(2)} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(4)} h_{(4)} b_{(1)}^{(\bar{2})}{}_{(4)} d_{(\bar{4})}^{(\bar{1})}{}_{(2)} d_{(\bar{5})}^{(\bar{1})}{}_{(2)}) (b_{(2)} \triangleleft d_{(\bar{5})}^{(\bar{1})}{}_{(3)}) \langle c_{(\bar{1})}, a_{(1)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{3})}^{(\bar{2})}, a_{(5)} \rangle \\
&\quad \langle d_{(\bar{2})}, a_{(2)}^{(\bar{1})} \triangleleft c_{(\bar{3})}^{(\bar{1})}{}_{(1)} h_{(1)} b_{(1)}^{(\bar{2})}{}_{(1)} \rangle \langle d_{(\bar{1})}, b_{(1)}^{(\bar{1})} \rangle \langle \bar{S} d_{(\bar{4})}^{(\bar{2})}, a_{(4)} \triangleleft b_{(1)}^{(\bar{2})}{}_{(5)} \rangle \langle \bar{S} d_{(\bar{5})}^{(\bar{2})}, b_{(3)} \rangle.
\end{aligned}$$

The calculation for  $a \cdot ((chb) \cdot d)$  is strictly similar: we use the second line of (61) and the other halves of (56)–(57). Indeed, there is a strict symmetry involving: reversal of all products and

tensor products (reflection in a mirror), interchange of  $a, b, \triangleright$  with  $d, c, \triangleleft$ , interchange of  $\langle \cdot, \cdot \rangle$  with  $\langle \bar{S} \cdot, \cdot \rangle$  and reversal of the numbering of the coactions and all coproducts. On the other hand, our final expression for  $(a \cdot (chb)) \cdot d$  is self-symmetric under this operation. Hence it coincides with the result for  $a \cdot ((chb) \cdot d)$ . Hence associativity is proven.

For the coalgebra structure, our requirement that  $H \bowtie B$  and  $\bar{C} \bowtie H$  are sub-bialgebras forces the general coproduct to be

$$\Delta(chb) = c_{(\bar{1})} \otimes c_{(\bar{2})}^{(\bar{1})} h_{(1)} \otimes \underline{b}_{(1)}^{(\bar{1})} \otimes c_{(\bar{2})}^{(\bar{2})} \otimes h_{(2)} \underline{b}_{(1)}^{(\bar{2})} \otimes \underline{b}_{(2)}. \quad (62)$$

As before, we only need to prove the bialgebra homomorphism property for the special case  $\Delta(b \cdot c)$ . For brevity, we outline the proof using the cross relations stated in the theorem rather than the more explicit ordering relations (59). This is equivalent, although less direct. Thus, working in the algebra  $\mathcal{U}$ , we have

$$\begin{aligned} (\Delta b)(\Delta c) &= \underline{b}_{(1)}^{(\bar{1})} c_{(\bar{1})} c_{(\bar{2})}^{(\bar{1})} \otimes \underline{b}_{(1)}^{(\bar{2})} \underline{b}_{(2)} c_{(\bar{2})}^{(\bar{2})} \\ &= \underline{b}_{(1)}^{(\bar{2})} c_{(\bar{1})} c_{(\bar{2})}^{(\bar{1})} \underline{b}_{(2)}^{(\bar{1})} c_{(\bar{3})}^{(\bar{1})} c_{(\bar{4})}^{(\bar{1})} c_{(\bar{5})}^{(\bar{1})} c_{(\bar{6})}^{(\bar{1})} c_{(\bar{1})}^{(\bar{1})} \otimes \underline{b}_{(1)}^{(\bar{2})} c_{(\bar{2})}^{(\bar{2})} \underline{b}_{(2)}^{(\bar{2})} \underline{b}_{(3)}^{(\bar{2})} \underline{b}_{(4)}^{(\bar{2})} c_{(\bar{5})}^{(\bar{2})} \underline{b}_{(5)} c_{(\bar{6})}^{(\bar{1})} c_{(\bar{2})}^{(\bar{2})} \\ &\quad \langle c_{(\bar{1})}, \underline{b}_{(1)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{3})}^{(\bar{2})}, \underline{b}_{(3)}^{(\bar{1})} \rangle \langle c_{(\bar{4})}^{(\bar{2})}, \underline{b}_{(4)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{6})}^{(\bar{2})}, \underline{b}_{(6)} \rangle \\ &= \underline{b}_{(1)}^{(\bar{2})} c_{(\bar{1})} c_{(\bar{2})}^{(\bar{1})} \underline{b}_{(2)}^{(\bar{1})} c_{(\bar{3})}^{(\bar{1})} c_{(\bar{4})}^{(\bar{1})} c_{(\bar{5})}^{(\bar{1})} c_{(\bar{6})}^{(\bar{1})} c_{(\bar{1})}^{(\bar{1})} \otimes \underline{b}_{(1)}^{(\bar{2})} c_{(\bar{2})}^{(\bar{2})} \underline{b}_{(2)}^{(\bar{2})} \underline{b}_{(3)}^{(\bar{2})} c_{(\bar{4})}^{(\bar{2})} \underline{b}_{(4)} c_{(\bar{5})}^{(\bar{1})} c_{(\bar{2})}^{(\bar{2})} \\ &\quad \langle c_{(\bar{1})}, \underline{b}_{(1)}^{(\bar{1})} \rangle \langle (\bar{S} c_{(\bar{3})}^{(\bar{2})}) c_{(\bar{3})}^{(\bar{2})}, \underline{b}_{(3)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{5})}^{(\bar{2})}, \underline{b}_{(5)} \rangle \\ &= \underline{b}_{(1)}^{(\bar{2})} c_{(\bar{1})} c_{(\bar{2})}^{(\bar{1})} \underline{b}_{(2)}^{(\bar{1})} c_{(\bar{3})}^{(\bar{1})} c_{(\bar{4})}^{(\bar{1})} c_{(\bar{5})}^{(\bar{1})} c_{(\bar{6})}^{(\bar{1})} c_{(\bar{1})}^{(\bar{1})} \otimes \underline{b}_{(1)}^{(\bar{2})} c_{(\bar{2})}^{(\bar{2})} \underline{b}_{(2)}^{(\bar{2})} c_{(\bar{3})}^{(\bar{2})} \underline{b}_{(3)} c_{(\bar{4})}^{(\bar{1})} c_{(\bar{2})}^{(\bar{2})} \langle c_{(\bar{1})}, \underline{b}_{(1)}^{(\bar{1})} \rangle \langle \bar{S} c_{(\bar{4})}^{(\bar{2})}, \underline{b}_{(4)} \rangle \\ &= \Delta(bc), \end{aligned}$$

where the second equality uses the cross relations in each factor of  $\mathcal{U} \otimes \mathcal{U}$  and covariance of the braided coproducts under the coaction of  $H$ . The third equality uses the pairing axiom (56) with  $\underline{b}_{(3)}^{(\bar{1})} \otimes \underline{b}_{(4)}^{(\bar{1})} \otimes \underline{b}_{(3)}^{(\bar{2})} \underline{b}_{(4)}^{(\bar{2})} = \underline{\Delta}(\underline{b}_{(3)}^{(\bar{1})}) \otimes \underline{b}_{(3)}^{(\bar{2})}$ . We then use the braided-antipode property in  $\bar{C}$ . Finally we recognise  $\Delta(bc)$  in using the further identity

$$b^{(\bar{1})} c^{(\bar{1})} \otimes b^{(\bar{2})} c^{(\bar{2})} = c^{(\bar{1})} b^{(\bar{1})} \otimes c^{(\bar{2})} b^{(\bar{2})}, \quad \forall b \in B, c \in \bar{C}, \quad (63)$$

which follows directly from (58). As before, the antipode on  $\mathcal{U}$  exists and is uniquely determined by the antipodes of  $H \bowtie B$  and  $\bar{C} \bowtie H$ .  $\square$

We remark that we do not need the antipode or inverse antipode of  $H$  for this construction, though this is the natural setting for the input data. Without the inverse antipodes, the braidings

$\Psi$  in (55) are not invertible, but the braided-homomorphism properties as in (61) still make sense. We do not need braided antipodes on  $B, \bar{C}$  either, but only need to assume a ‘convolution inverse’ to  $\langle \cdot, \cdot \rangle$  in place of  $\langle \bar{S}(\cdot), \cdot \rangle$ , as characterised by (56)–(57). In this way, the above construction lifts entirely to the bialgebra level. We can also write the relations in the theorem as

$$b_{(1)}c_{(\bar{1})}c_{(\bar{2})}^{(\bar{1})}\langle c_{(\bar{2})}^{(\bar{2})}, b_{(2)} \rangle = \langle c_{(\bar{1})}, b_{(1)}^{(\bar{1})} \rangle b_{(1)}^{(\bar{2})}c_{(\bar{2})}b_{(2)}. \quad (64)$$

This is an extension of the construction of Section 3. The precise inclusion of one construction in the other is provided by the functors  ${}_H\mathcal{M} \hookrightarrow {}^H_H\mathcal{M}$  and  $\mathcal{M}_H \hookrightarrow \mathcal{M}_H^H$  when  $H$  is quasitriangular. These functors (introduced by the author in [24]) use the quasitriangular structure to induce from an action a compatible coaction, forming a crossed module. In the specific setting of Section 3 we have  $B \in \mathcal{M}_H \hookrightarrow \mathcal{M}_H^H$  and  $\bar{C} \in {}_{\bar{H}}\mathcal{M} \hookrightarrow {}^{\bar{H}}_{\bar{H}}\mathcal{M}$  by induced right, left coactions

$$b^{(\bar{1})} \otimes b^{(\bar{2})} = b \triangleleft \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)}, \quad c^{(\bar{1})} \otimes c^{(\bar{2})} = \mathcal{R}^{-(1)} \otimes \mathcal{R}^{-(2)} \triangleright c \quad (65)$$

respectively. In the weak quasitriangular case we have  $B \in {}^A\mathcal{M}$  and define both the action of  $H$  and the induced coaction of  $H$  from this data as explained in Remark 3.9 (i.e. the action is by evaluation against the coaction of  $A$ , and the induced coaction is  $b \mapsto b^{(\bar{2})} \otimes \mathcal{R}(b^{(\bar{1})})$ ). Similarly for  $\bar{C}$ . These inclusions are the standard way that bosonisations can be viewed as examples of biproducts. It is easy to see that the condition (58) also holds by cancellation of  $\mathcal{R}, \mathcal{R}^{-1}$ , so the double-bosonisation  $U$  in Section 3 can be viewed as an example of the more general  $\mathcal{U}$  above.

The double biproduct construction includes other examples as well. Thus, the dual bosonisation construction is obviously also a biproduct (see the Preliminaries), this time using the dual quasitriangular structure  $\mathcal{R} : H \otimes H \rightarrow k$  to induce an action from a coaction. In the present case we can use these to map braided groups  $B \in \mathcal{M}_H \hookrightarrow \mathcal{M}_H^H$  and  $\bar{C} \in {}_{\bar{H}}\mathcal{M} \hookrightarrow {}^{\bar{H}}_{\bar{H}}\mathcal{M}$  by

$$b \triangleleft h = b^{(\bar{1})} \mathcal{R}(b^{(\bar{2})} \otimes h), \quad h \triangleright c = \mathcal{R}^{-1}(h \otimes c^{(\bar{1})})c^{(\bar{2})} \quad (66)$$

respectively. Here  $\bar{H}$  denotes  $H$  with the inverse-transpose dual quasitriangular structure.

**Example B.2** *Let  $G$  be an Abelian group,  $\beta : G \times G \rightarrow k$  an invertible symmetric bicharacter and  $B, D$  two  $G$ -graded Hopf algebras dually-paired in the usual way (respecting the grading). Let  $\bar{C} = D^{\text{cop}}$ . Then there is a double-biproduct  $\mathcal{U} = \mathcal{U}(\bar{C}, kG, B)$ . For any  $g \in G$  and  $b, c$*

braided-primitive of homogeneous  $G$ -degree  $| \cdot |$ ,  $\mathcal{U}$  has the structure

$$\begin{aligned} g^{-1}cg &= \beta(g, |c|)c, & g^{-1}bg &= \beta(|b|, g)b, & [b, c] &= (|b| - |b|^{-1})\langle c, b \rangle \\ \Delta g &= g \otimes g, & \Delta b &= b \otimes |b| + 1 \otimes b, & \Delta c &= c \otimes 1 + |c| \otimes c \\ \epsilon g &= 1, & \epsilon b &= \epsilon c = 0, & Sg &= g^{-1}, & Sb &= -b|b|^{-1}, & Sc &= -|c|^{-1}c. \end{aligned}$$

**Proof** It is well known that group-graded algebras can be considered as  $kG$ -comodule algebras, where  $kG$  is the group algebra of  $G$ , see e.g. [35].  $G$ -graded Hopf algebras can be considered in the same way if there is a bicharacter  $\beta$ , which we extend by linearity to a dual-quasitriangular structure on  $H = kG$  cf.[13][28].  $G$ -graded Hopf algebras thereby become Hopf algebras in the braided category of comodules of a dual-quasitriangular Hopf algebra as introduced (by the author) in [2]. We suppose that  $B, D$  are such  $G$ -graded braided groups, equipped with an ordinary pairing  $\langle \cdot, \cdot \rangle : D \otimes B \rightarrow k$  preserving the degree in the sense  $\langle c, b \rangle |b| = |c|^{-1} \langle c, b \rangle$  on all homogeneous elements  $b \in B, c \in D$ . We view  $B$  in the category of right  $kG$ -comodules by  $b \mapsto b \otimes |b|$ , and  $D$  in the category of left  $kG$ -comodules by  $c \mapsto |c| \otimes c$ . Finally, we let  $\bar{C}$  be the same braided group as  $D$  but with braided-opposite coproduct  $\bar{\Delta}$ . Then  $\bar{C}$  lives in the braided category of left  $kG$ -comodules with braiding determined by the inverse-transposed bicharacter. Explicitly,

$$\underline{\Delta}(ab) = \beta(|a_{(2)}|, |b_{(1)}|)a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}, \quad \bar{\Delta}(cd) = \beta^{-1}(|c_{(2)}|, |d_{(1)}|)c_{(1)}d_{(1)} \otimes c_{(2)}d_{(2)}$$

on homogeneously decomposed coproducts. As in (66) we view  $B, \bar{C}$  in the right/left crossed  $kG$ -module categories via the induced actions  $b \triangleleft g = b\beta(|b|, g)$  and  $g \triangleright c = \beta^{-1}(g, |c|)c$ . It is easy to see that (56) are satisfied; the bicovariance of the pairing requiring the symmetry of  $\beta$ . We then apply Theorem B.1 and compute the structure of  $\mathcal{U}$  as follows: the  $H \bowtie B$  and  $\bar{C} \bowtie H$  algebras are the corresponding right and left cross products as shown, while the cross relations (64) and coproduct (62) become

$$b_{(1)}c_{(1)}|c_{(2)}|\langle c_{(2)}, b_{(2)} \rangle = \langle c_{(1)}, b_{(1)} \rangle |b_{(1)}|c_{(2)}b_{(2)}, \quad \Delta b = b_{(1)} \otimes |b_{(1)}|b_{(2)}, \quad \Delta c = c_{(1)}|c_{(2)}| \otimes c_{(2)}, \quad (67)$$

which computes as stated on braided-primitive elements.  $\square$

The same construction works if  $\beta$  is skew-symmetric as in [36]. The only difference is that in this case we suppose that  $\langle \cdot, \cdot \rangle$  preserves the grading in the sense  $\langle c, b \rangle |b| = |c| \langle c, b \rangle$  (no



inversion). This case is not so interesting, however, because on braided-primitive elements one then has  $[b, c] = 0$ . Both cases work more generally; the skew case works with  $kG$  replaced by any triangular Hopf algebra  $H$ . We just consider  $D \in {}^H\mathcal{M}$  dually paired with  $B \in \mathcal{M}^H$  in a covariant manner (in the sense  $c^{(\bar{1})}\langle c^{(\bar{2})}, b \rangle = \langle c, b^{(\bar{1})} \rangle b^{(\bar{2})}$ ), and set  $\bar{C} = D^{\text{cop}} \in {}^H\mathcal{M}$ . However, we again have  $[b, c] = 0$  on braided-primitive elements.

It seems likely that there is variant of Theorem B.1 which works for general dual quasitriangular Hopf algebras too. The present version is enough to include Example B.2, which generalises Lusztig's construction in Section 4 to the case of a symmetric bicharacter on any (not-necessarily free) Abelian group  $G$  and a pair of suitably dual  $G$ -graded braided groups. Note however, that there is in general no surjection to  $\mathcal{U}$  from the quantum double as in the preceding Appendix A, and hence no corresponding quasitriangular structure in the finite-dimensional non-degenerately paired case. As with single bosonisations and their duals, the key properties of  $U$  in Section 3 do not come from this biproduct point of view, though it may be useful as a complement.

## C Appendix: Diagrammatic construction of $U$

In this Appendix we provide a more category-theoretic point of view on the  $U(B)$  construction in Section 3: we construct a braided category and recover  $U$  by Tannaka-Krein reconstruction as its generating Hopf algebra. The construction is more general than the algebraic one in Section 3, though this remains the main example.

We work in a general braided monoidal or quasitensor category  $\mathcal{C}$ . Let  $B$  be a braided group (Hopf algebra) in  $\mathcal{C}$  and let  $C$  be another braided group in  $\mathcal{C}$  which is dually paired with  $B$  from the left, i.e. there is a morphism  $\text{ev} : C \otimes B \rightarrow \underline{1}$  obeying the categorical duality axioms as explained in the Preliminaries. Here  $\underline{1}$  denotes the identity object in  $\mathcal{C}$ . We use a diagrammatic notation in which braidings are denoted  $\Psi = \bowtie$ , inverse braidings by the reversed braid crossing,  $\text{ev} = \cup$  and  $\underline{1}$  is omitted. Other morphisms are represented by nodes with the appropriate valency, pointing generally downwards. Unless otherwise labelled,  $\cdot = \vee$  a product and  $\underline{\Delta} = \wedge$  a coproduct. We suppress the associativity morphisms in the category. Functoriality of the braiding says that morphisms on nodes can be pulled through braid crossings on either side. Algebra and Hopf algebra in braided categories using this notation was introduced by the author, in [1][7][6][4][3].

**Definition C.1** Let  $C, B$  be dually paired braided groups in  $\mathcal{C}$ . We define the category  ${}_C\mathcal{C}_B$  of braided crossed  $C - B$ -bimodules to consist of objects  $(V, \triangleright, \triangleleft)$  where  $V$  is an object of  $\mathcal{C}$ ,  $\triangleright$  is a left action in the category of  $C$  on  $V$  and  $\triangleleft$  is a right action of  $B$  on  $V$  such that

$$\begin{aligned} & (\triangleright \otimes \text{ev}) \circ (\text{id} \otimes \Psi_{C,V} \otimes \text{id}) \circ (\text{id} \otimes \text{id} \otimes \triangleleft \otimes \text{id}) \circ (\Delta_C \otimes \text{id} \otimes \Delta_B) \\ &= (\text{ev} \otimes \triangleleft) \circ (\text{id} \otimes \Psi_{V,B} \otimes \text{id}) \circ (\text{id} \otimes \triangleright \otimes \text{id} \otimes \text{id}) \circ (\Delta_C \otimes \text{id} \otimes \Delta_B) \end{aligned} \quad (68)$$

as morphisms  $C \otimes V \otimes B \rightarrow V$ . The condition is shown in Figure 1(a). Morphisms in  ${}_C\mathcal{C}_B$  are morphisms in  $\mathcal{C}$  which intertwine the actions of  $C, B$ .

**Proposition C.2** The category  ${}_C\mathcal{C}_B$  is monoidal, where the tensor product is the usual tensor product of  $C$  and  $B$  modules in  $\mathcal{C}$ . The forgetful functor  ${}_C\mathcal{C}_B \rightarrow \mathcal{C}$  is monoidal.

**Proof** The proof is shown in Figure 1(c). The left hand diagram is the condition (68) from Figure 1(a) applied to the module  $V \otimes W$ . The latter is a right  $B$  module via the coproduct of  $B$  and the braiding  $\Psi$  of  $\mathcal{C}$ , and a left  $C$ -module via the coproduct of  $C$  in the usual way[4]. Iterated coproducts  $(\text{id} \otimes \underline{\Delta}) \circ \underline{\Delta} = (\underline{\Delta} \otimes \text{id}) \circ \underline{\Delta}$  are depicted by nodes with 1 input line and 3 output lines. The first identity applies the assumed condition (68) for  $W$  (the upper layer in the figure). The second identity then applies the assumed condition (68) for  $V$  (the lower layer). The result is the right hand side of Figure 1(a) in for the tensor product module  $V \otimes W$ . The identity object  $\underline{1}$  equipped with the trivial module structures via the counits of  $C, B$  provides the identity object of  ${}_C\mathcal{C}_B$ . Since we build the tensor product on  $V \otimes W$ , the forgetful functor of  $\mathcal{C}$  is monoidal.  $\square$

If there is also a coevaluation  $\text{coev} : \underline{1} \rightarrow B \otimes C$  making  $B$  a rigid object in the category, we write  $C = B^*$  and  $\text{coev} = \cap$ . The ‘bend-straightening axioms’ pertain, see [3]. In this case we can turn right  $B^*$ -modules into right  $B$ -comodules in  $\mathcal{C}$ . Then  ${}_{B^*}\mathcal{C}_B$  is equivalent to the category of crossed  $B$ -modules  ${}^B\mathcal{C}_B$  as in [17][18], where the latter category is shown to be braided. In our formulation the same observation is:

**Proposition C.3** cf.[17][18]  ${}_{B^*}\mathcal{C}_B$  is braided by

$$\Psi_{(V, \triangleleft, \triangleright), (W, \triangleleft, \triangleright)} = \Psi_{V,W} \circ (\triangleleft \otimes \triangleright) \circ (\text{id} \otimes S \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \text{coev} \otimes \text{id})$$

as shown in Figure 1(b).

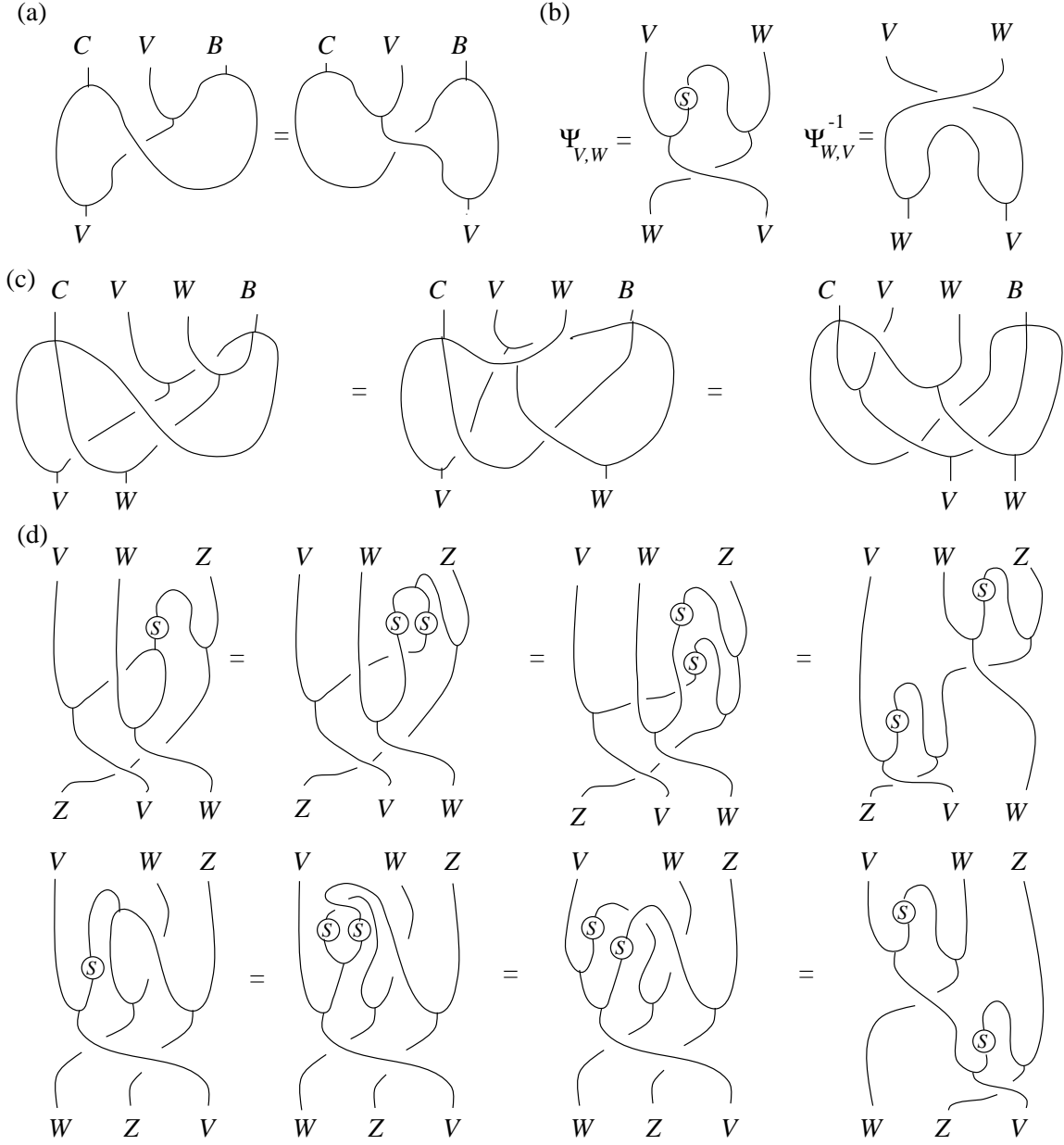


Figure 1: Compatibility condition (a) defining the category of  $C - B$ -crossed bimodules. Its braiding (b) in the rigid case. Proof (c) that the category is monoidal and proof (d) that the braiding obeys hexagon identities

**Proof** This is shown in Figure 1(d). The upper left hand side shows  $\Psi_{(V \otimes W, \triangleleft, \triangleright), (Z, \triangleleft, \triangleright)}$  where we use the tensor product action on  $V \otimes W$  from Proposition C.2. The first equality is the braided-antimultiplicativity of  $S$  proven in [6]. The second identity dualises the coproduct of  $B$  as a product in  $B^*$  and then writes the action by this product as two applications of  $\triangleright$ . The third identity uses functoriality to write the result as  $\Psi_{(V, \triangleleft, \triangleright), (Z, \triangleleft, \triangleright)} \circ \Psi_{(W, \triangleleft, \triangleright), (Z, \triangleleft, \triangleright)}$ . This verifies one of the so-called hexagon identities for the braiding in  ${}_B\mathcal{C}_B$ . The second line in Figure 1(d) is the proof of the other hexagon, and is similar. The first identity dualises the coproduct in  $B^*$  as a product in  $B$  and applies braided-antimultiplicativity of the braided antipode. The second identity uses that  $\triangleleft$  is an action. Functoriality then provides the required right hand side. The inverse braiding shown in Figure 1(b) is clearly the inverse morphism after we use that  $\triangleleft, \triangleright$  are actions to write the composition of  $\Psi, \Psi^{-1}$  as the action of products in  $B^*, B$ . The product in  $B^*$  can then be written as the coproduct in  $B$ , providing an antipode loop, which we cancel (by the axiom of a braided antipode).  $\square$

These steps are similar to the study of the representations of the usual quantum double in [24], except that now all modules are objects in a background braided category. Just as the quantum double has a canonical ‘Schrödinger representation’ by the coregular and adjoint actions, we show now that the same holds in our braided setting.

**Proposition C.4** *Let  $C, B$  be dually paired braided groups in  $\mathcal{C}$ . Then  $V = B$  is an algebra in  ${}_C\mathcal{C}_B$  where  $B$  acts on itself by the right adjoint action[15] (upper box in Figure 2) and  $C$  acts on  $B$  by the left coregular representation[3][8] (lower box in Figure 2).*

**Proof** This is shown in Figure 2. The right adjoint action shown is the mirror image of the left adjoint action in [15]. More precisely, consider the proofs for the left adjoint action of a braided group on itself, reflect in a mirror about a vertical axis and reverse all braid crossings. This gives the proof that the  $B$  is a right  $B$ -module algebra by the right-handed Ad as shown. The left coregular representation shown in the lower box is the right coaction provided by the coproduct, converted to a left action of  $C$  via the evaluation pairing (a coevaluation is not required)[4]. Equivalently, it is the left braided differentiation in [14] provided by evaluation against the left output of the coproduct, but applied to  $B^{\text{cop}}$  with coproduct  $\Psi^{-1} \circ \underline{\Delta}$ . The

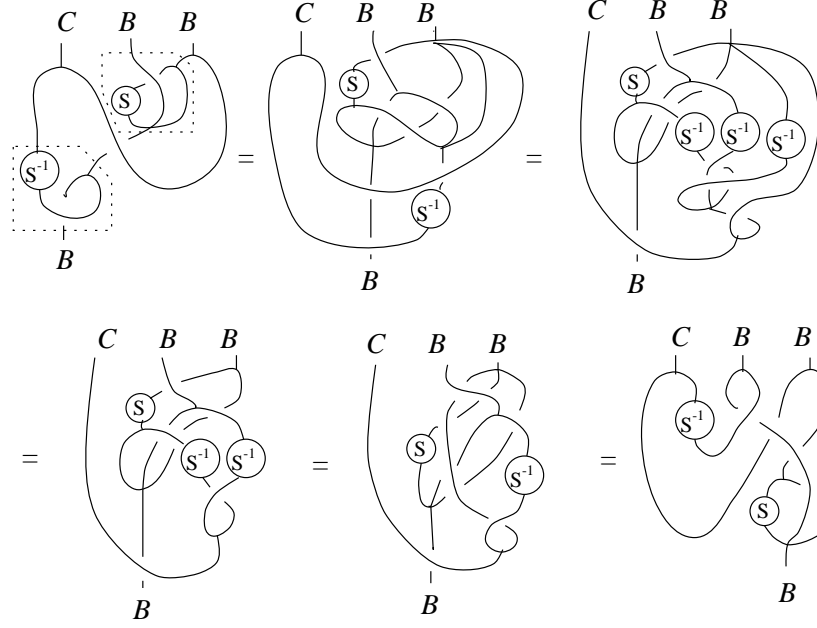


Figure 2: Proof that  $B$  itself is an object in the category of  $C - B$ -crossed bimodules

braided antipode converts  $B$  to a braided left  $C$ -module algebra according to [4]. Hence  $V = B$  is a right  $B$ -module algebra in  $\mathcal{C}$  and a left  $C$ -module algebra in  $\mathcal{C}$ .

It remains to show that the left and right actions obey (68) in Figure 1(a). The first expression in Figure 2 is the left hand side of this condition. The first equality is uses the coproduct homomorphism property to compute the coproduct of a triple-product. We combine iterated coproducts into a multiple node. The second identity is the braided-antimultiplicativity of the inverse braided antipode[3], iterated. We also write the coproduct of  $C$  as a product in  $B$ , in view of their duality pairing. The third equality cancels a loop involving the inverse braided antipode. The latter is the antipode for  $B^{\text{op}}$ [3]. The fourth equality uses the braided-antimultiplicativity of  $\underline{S}$  to simplify further. The last equality writes a product and inverse braided antipode of  $B$  back in terms of the coproduct and inverse braided-antipode of  $C$ . We obtain the right hand side of the condition (68).  $\square$

The corresponding result for braided crossed modules is also new: the braided group  $B$  provides a canonical algebra in  ${}^B\mathcal{C}_B$  by the right adjoint action and the coproduct viewed as a left  $B$ -comodule in the category.

These are general categorical constructions. We now apply them in the case  $\mathcal{C} = \mathcal{M}_H$  where  $H$  is a quasitriangular Hopf algebra with quasitriangular structure  $\mathcal{R}$ . Similarly in  ${}^A\mathcal{M}$  for a weakly quasitriangular Hopf algebra dual pair  $(H, A, \mathcal{R})$ . We consider the *completely forgetful functor* which is the forgetful functor  $\mathcal{M}_H \rightarrow \text{Vec}$  composed with the forgetful functor  ${}_C\mathcal{C}_B \rightarrow \mathcal{M}_H$ . It is manifestly monoidal and hence, by general Tannaka-Krein arguments we deduce the existence of a Hopf algebra through the (say, right) modules of which it factors. In the present setting we have an equivalence of categories:

**Proposition C.5** *Let  $C, B$  be dually paired braided groups in  $\mathcal{M}_H$ . Then the category  ${}_C\mathcal{C}_B$  of crossed  $C - B$ -modules is monoidally equivalent (in a way compatible with the forgetful functors to  $\text{Vec}$ ) to the category of right  $U$ -modules. If  $B$  is rigid then the equivalence becomes one of braided categories.*

**Proof** The calculation has been done in the proof of Lemma 3.11, where we used the concrete version of the diagrammatic condition (68), obtained from the form of braiding (4) in the category  $\mathcal{M}_H$ . We have seen that the left action of  $C$  can be viewed as a right action of  $\bar{C}$ , in which case the condition becomes the relation (24) in  $U$ . The  $\bar{C} \bowtie \bar{H}$  relations and  $H \bowtie B$  relations are just the condition that  $\triangleright, \triangleleft$  are morphisms in  $\mathcal{M}_H$ , as explained in the course of Lemma 3.11. If we know that these relations fully characterise  $U$ , as we know from Section 3 (see Remark 3.7), we conclude the result.  $\square$

The fundamental representation in Proposition C.5 provides in the case of  $\mathcal{M}_H$ , the concrete representation of  $U(\bar{C}, H, B)$  on  $B$  in Theorem 3.12. We read off the formulae there from the braiding (4) in  $\mathcal{M}_H$ . It is also possible to understand in categorical terms the projection  $\pi$  from the quantum double in Appendix A.

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